Combinatorial and Algebraic conditions that preclude SAPpiness

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Zero-nonzero Patterns

An $n \times n$ zero-nonzero pattern $\mathcal{A}$ is an $n \times n$ matrix with entries from the set $\{0, *\}$.

$$\mathcal{A} = \begin{bmatrix} * & * & 0 & 0 & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & * \\ * & * & 0 & 0 & * \end{bmatrix}$$

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An $n \times n$ zero-nonzero pattern $\mathcal{A}$ is an $n \times n$ matrix with entries from the set $\{0, *\}$.In order to better understand the nature of a zero-nonzero pattern we will often replace the $*$ entries with distinct variables. For Example

$$\mathcal{A} = \begin{bmatrix} * & * & 0 & 0 & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & * \\ * & * & 0 & 0 & * \end{bmatrix} \qquad A = \begin{bmatrix} a & b & 0 & 0 & c \\ 0 & 0 & d & 0 & e \\ 0 & 0 & 0 & f & 0 \\ 0 & 0 & g & 0 & h \\ j & k & 0 & 0 & \ell \end{bmatrix}$$

Spectrally arbitrary patterns

A zero-nonzero pattern $\mathcal{A}$ is called spectrally arbitrary, a SAP, if for each complex monic polynomial of degree n, $r(x)$, there exists a realization of $\mathcal{A}$ that has $r(x)$ as its characteristic polynomial.
For a Non-Example:

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$$\begin{bmatrix} * & * & * \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & 0 & 0 \\ e & f & 0 \end{bmatrix}$$

is not a SAP, since the characteristic polynomial will be $x^3 - ax^2 + aec$, and $a \neq 0$.

Digraph

The directed graph associated with an $n \times n$ zero-nonzero pattern $\mathcal{A}$ has vertex set $\{1, 2, \ldots, n\}$ and an arc directed from vertex i to vertex j if $\mathcal{A}_{i,j} = *$. For example

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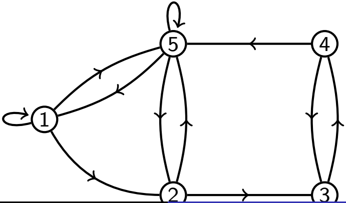

The Digraph and the Characteristic Polynomial

The digraph of a zero-nonzero pattern can be used to calculate the coefficients of the characteristic polynomial. Consider

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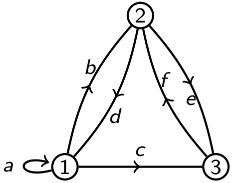

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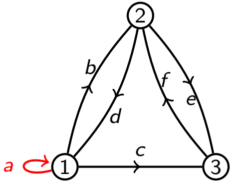

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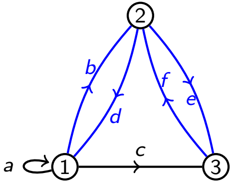

$$r(x) = x^3 + (-a)x^2 + (-bd - ef)x$$

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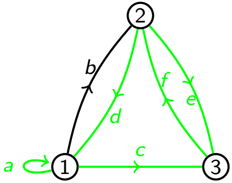

$$r(x) = x^3 + (-a)x^2 + (-bd - ef)x + aef - cdf$$

Known results

It is not hard to show that if $\mathcal{A}$ is a SAP then it has at least two nonzero diagonal entries, or in the digraph at least two loops.

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It is not hard to show that if $\mathcal{A}$ is a SAP then it has at least two nonzero diagonal entries, or in the digraph at least two loops. The generalization of this is that there are at least two covers of k vertices for all $k = 1, \ldots, n$.

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We set out to find combinatorial arguments that preclude a pattern from being a SAP. Consider the following digraph and associated zero-nonzero pattern.

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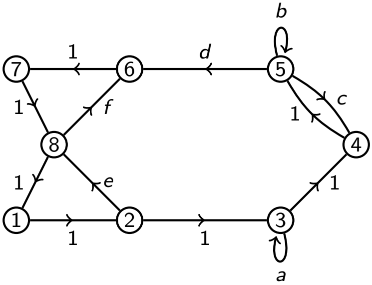

First result

Its characteristic polynomial has coefficients:

$$\begin{aligned}
a_1 &= -a - b \\
a_2 &= ab - c \\
a_3 &= ac - e - f = ac - (e + f) \\
a_4 &= ae + be + af + bf = (a + b)(e + f) \\
a_5 &= -abe - abf + ce + cf = (c - ab)(e + f) \\
a_6 &= -ace - acf = -ac(e + f) \\
a_7 &= 0 \\
a_8 &= -d
\end{aligned}$$

and each time e appears f appears with the same coefficient, so replacing $e + f$ with a new variable g we go from 6 variables to 5.

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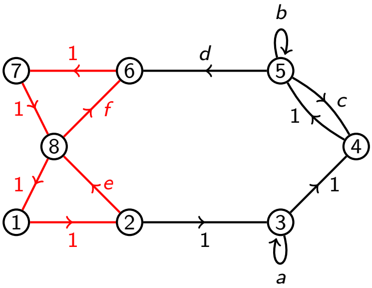

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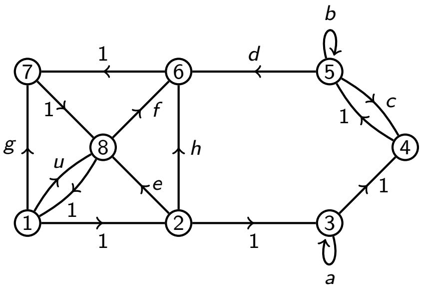

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The characteristic polynomial has coefficients:

$$\begin{aligned}
a_1 &= -a - b \\
a_2 &= ab - c - u \\
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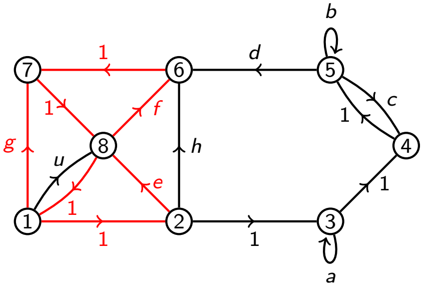

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a_6 &= ah + bh - ac(e + f + g) \\
a_7 &= -abh + ch \\
a_8 &= -ach - d
\end{aligned}$$

Replace $e + f + g$ with a new variable x we reduce the number of variables from 9 to 7. This had the potential to be a SAP with 16 nonzero entries it is brought down to $14 < 2n - 1$.

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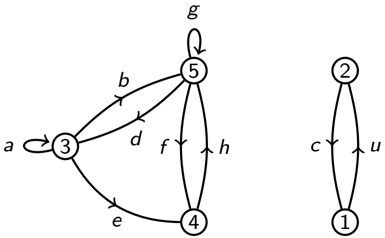
Then a "distinguished" edge can be chosen from each cycle in α such that the sum of the weights of the distinguished edges completely determines their contribution to the characteristic polynomial coefficients. In particular, this implies a loss of $m - 1$ algebraic "degrees of freedom."

Our next result shows that a disconnected digraph is not spectrally arbitrary and relies on it having a special form.

An illustrative example



The digraph corresponds to the following matrix, with nonzero entries replaced with variables.

$$\begin{bmatrix} 0 & u & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 \\ 0 & 0 & a & e & b \\ 0 & 0 & 0 & 0 & h \\ 0 & 0 & d & f & g \end{bmatrix}$$

Example Continued

The characteristic polynomial can be decomposed into the product of the two characteristic polynomials associated with its pieces, in this case:

$$x^5 + \alpha_1 x^4 + \alpha_2 x^3 + \alpha_3 x^2 + \alpha_4 x + \alpha_5 = (x^2 - cu)(x^3 + \overline{\alpha}_1 x^2 + \overline{\alpha}_2 x + \overline{\alpha}_3)$$

where

$$\begin{aligned} \overline{\alpha}_1 &= -a - g \\ \overline{\alpha}_2 &= ag - bd - fh \\ \overline{\alpha}_3 &= afh - beh \end{aligned}$$

Example continued

$$x^5 + \alpha_1 x^4 + \alpha_2 x^3 + \alpha_3 x^2 + \alpha_4 x + \alpha_5 = (x^2 - cu)(x^3 + \overline{\alpha}_1 x^2 + \overline{\alpha}_2 x + \overline{\alpha}_3)$$

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$$\begin{aligned} \alpha_1 &= \overline{\alpha}_1 \\ \alpha_2 &= \overline{\alpha}_2 - cu \\ \alpha_3 &= \overline{\alpha}_3 - cu\overline{\alpha}_1 \end{aligned}$$

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$$\begin{aligned} \alpha_1 &= \overline{\alpha}_1 \\ \alpha_2 &= \overline{\alpha}_2 - cu \\ \alpha_3 &= \overline{\alpha}_3 - cu\overline{\alpha}_1 \\ \alpha_4 &= -cu\overline{\alpha}_2 \end{aligned}$$

Example continued

$$x^5 + \alpha_1 x^4 + \alpha_2 x^3 + \alpha_3 x^2 + \alpha_4 x + \alpha_5 = (x^2 - cu)(x^3 + \overline{\alpha}_1 x^2 + \overline{\alpha}_2 x + \overline{\alpha}_3)$$

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\alpha_5 &= -cu\overline{\alpha}_3
\end{aligned}$$

Example Continued

This means that $\alpha_5 = -cu(\alpha_3 + cu(\alpha_1))$ and so if $\alpha_1 = \alpha_3 = 0$, then $\alpha_5 = 0$ as well.

Other cycles

In our example above we used a two cycle along with the “odd” coefficients, we could just as easily use a k-cycle and the coefficients that are $1, 2, \ldots, k-1$ modulo k. The result will be the same.

A bit of Algebra

We use the algebraic structure of $\mathbb{C}[\vec{x}]$ to make some statements about the coefficients of the Characteristic Polynomial.

In particular we think about when one of the coefficients is a multiple of another. In the case that $\alpha_j = c\alpha_i$, then $\alpha_i = 0$ implies $\alpha_j = 0$ and the pattern is not a SAP.

Divisibility

Definition

Suppose $1 \leq i < j \leq n$. We say that the j-covers are *completely symmetric* with respect to the i-covers if the following two conditions hold

1. Each j-cover contains a unique i-cover.
2. Within each j-cover, the i-cover it contains can be "swapped out" for any other i-cover to create a different j-cover.

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Theorem

Suppose $i < j$. The j-covers are completely symmetric with respect to the i-covers if and only if α_i divides α_j.

Beyond divisibility

We notice that the fact that α_j is a multiple of α_i can be thought of algebraically as $\alpha_j \in \langle \alpha_i \rangle$. Where $\langle \alpha_i \rangle$ is the ideal generated by the coefficient α_i.

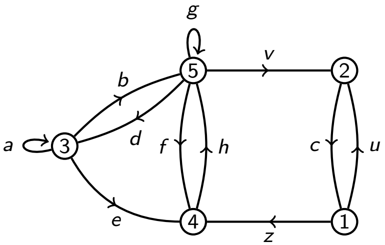
A natural extension of this would be the question, when is $\alpha_j \in \langle \alpha_i : i \neq j \rangle$. We have already seen an example of this in our disconnected graph.

Beyond Divisibility

A final extension of this divisibility might be, since each of the variables in our pattern are nonzero, is there a monomial of variables in the ideal? In particular we want to know if $(\prod_j x_j)^m \in \langle \alpha_i \rangle$.

Example

Taking the disconnected example above and adding arcs, $(1,4)$ and $(5,2)$ gives rise to



Example

The only cycle covers added by this addition of arcs are weighted *chvz* in α_4 and *achvz* in α_5. In particular the coefficients of the characteristic polynomial are

$$\begin{aligned}
\alpha_1 &= -a - g \\
\alpha_2 &= -bd + ag - fh - cu \\
\alpha_3 &= -deh + afh + acu + gcu \\
\alpha_4 &= bcdu - acgu + cfhu - chvz \\
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Since α_1 and α_3 are unchanged we have that the monomial $achvz \in \langle \alpha_1, \alpha_3, \alpha_5 \rangle$ and so the pattern is not spectrally arbitrary.

Beyond Monomials

It would be bad if $\left(x_1 x_2 \cdots x_r \cdot \prod_{k \in S} \alpha_k\right)^m \in \langle \alpha_j : j \notin S \rangle$

This condition can be tested using a computer.

$2n$ conjecture Computer Search

Idea: For a given n use a computer to check each irreducible pattern with $2n-1$ nonzero entries against this algebraic condition.

For $n = 3, 4, 5, 6$ all of the patterns with $2n - 1$ nonzero entries satisfy this algebraic condition and are therefore not spectrally arbitrary.

For $n = 7$

When $n = 7$ we are left with 5 patterns, only two of which need to be checked.

Mystery Pattern 1:

$$\begin{bmatrix} 0 & * & 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * & * & 0 \\ * & 0 & 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & 0 & 0 & * & 0 \end{bmatrix}$$

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Mystery Pattern 2:

$$\begin{bmatrix} 0 & * & 0 & 0 & * & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & * \\ 0 & * & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 \\ * & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 & 0 & 0 \end{bmatrix}$$

What's Known

For mystery pattern 2 there is a clear argument that if we are looking over $\mathbb{R}$, then the pattern is not spectrally arbitrary, i.e. some choice of coefficients makes one of the other coefficients the sum of squares and so must be positive over $\mathbb{R}$.

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For mystery pattern 1 Tracy Hall found a characteristic polynomial that is not attainable except by using Complex entries.

$$x^7 - 4x^6 + 27x^5 + 8x^4 - 24x^3 + 64x^2 + 192x$$

So the claim is that the $2n$ conjecture holds over $\mathbb{R}$ for $n = 7$.

Some Questions

- Can one algorithmically identify the combinatorial conditions given?
- Are the two mystery patterns spectrally arbitrary over $\mathbb{C}$?
- Is it possible to randomly select a pattern that does not satisfy the algebraic condition set forth with large enough n that is in fact spectrally arbitrary?
- What can we say about the $2n$ conjecture over $\mathbb{R}$ versus over $\mathbb{C}$?

Thank you for your attention, any questions?