UNIVERSITÉ LAVAL

Eigenvalues of Doubly-Stochastic Matrices
An Unfinished Story

Javad Mashreghi
Université Laval
(Joint work with R. Rivard)

Special W-CLAM
BIRS
July 8-9, 2017

Definition

A square matrix $A = [a_{ij}]$ is called stochastic if

1. $a_{ij} \geq 0$.
2. For each i, $\sum_{j=1}^{n} a_{ij} = 1$.

A Simple-looking Question

Fix $n \geq 2$. Which points λ in the complex plane can serve as the

eigenvalues

of an $n \times n$ stochastic matrix?

Kolmogorov's question (1938)

Let A be a stochastic matrix.
Put

$$\Omega_n = \{\lambda \in \mathbb{C} : \lambda \text{ is the eigenvalues of an } n \times n \text{ stochastic matrix}\}.$$

In other words, Ω_n is the collection of *all eigenvalues* of *all $n \times n$ stochastic matrices*.

Kolmogorov's question (1938)

Let A be a stochastic matrix.
Put

$$\Omega_n = \{\lambda \in \mathbb{C} : \lambda \text{ is the eigenvalues of an } n \times n \text{ stochastic matrix}\}.$$

In other words, Ω_n is the collection of *all eigenvalues* of *all $n \times n$ stochastic matrices*.

Question: What is Ω_n?

A Simple Observation

$$\Omega_n \subset \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}.$$

Justification

Let

$$\lambda \in \Omega_n.$$

Hence, there is an $n \times n$ stochastic matrix $A = [a_{ij}]$ and a non-zero n-dimensional vector $\overrightarrow{X} = (x_1, x_2, \ldots, x_n)^{tr}$ such that $A\overrightarrow{X} = \lambda \overrightarrow{X}$. Thus, for each i,

$$\lambda x_i = \sum_{j=1}^{n} a_{ij} x_j.$$

Justification

Choose the index i_0 such that

$$|x_{i_0}| = \|\overrightarrow{X}\|_\infty := \max\{|x_1|, |x_2|, \dots, |x_n|\}.$$

Surely, $x_{i_0} \neq 0$, otherwise $x_1 = \cdots = x_n = 0$, i.e.,

$$\overrightarrow{X} = 0,$$

which is a contradiction.

Justification

Then

$$
\begin{aligned}
|\lambda|\;|x_{i_0}| &= \left|\sum_{j=1}^{n} a_{ij}\, x_j\right| \\
&\le \sum_{j=1}^{n} a_{ij}\,|x_j| \\
&\le |x_{i_0}| \sum_{j=1}^{n} a_{ij} = |x_{i_0}|.
\end{aligned}
$$

Therefore, $|\lambda| \le 1$.

Another Simple Observation

Let $\overrightarrow{X} = (1, 1, \ldots, 1)^{tr}$. Then, for each stochastic matrix A,

$$A\overrightarrow{X} = \overrightarrow{X}.$$

Therefore,

$$1 \in \Omega_n.$$

History of Ω_n

1. Kolmogorov (1938): The question was raised.
2. Romanovsky (1936): $\Omega_n \cap \mathbb{T}$ was found.
3. Dmitriev et Dynkin (1945): Found the forbidden region around 1.
4. Dmitriev et Dynkin (1946): Found $\Omega_2, \ldots, \Omega_5$ and partial answer for $n \geq 6$.
5. Karpelevich (1951): Complete characterization (difficult formulas).
6. Ito (1997): Simplified formulas.

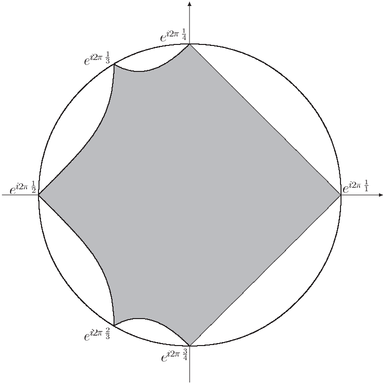


Figure: The region Ω_4.

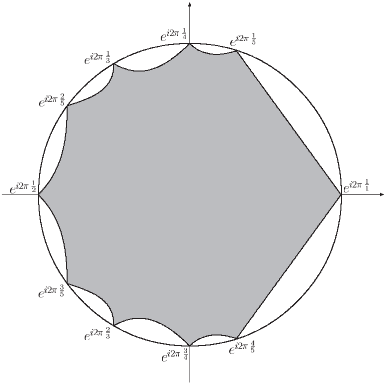


Figure: The region Ω_5.

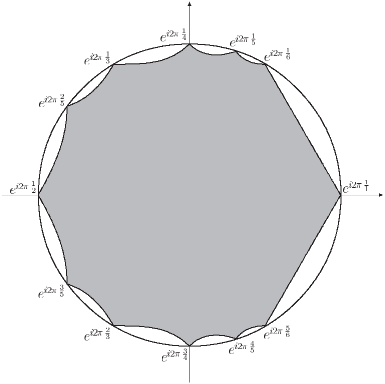


Figure: The region Ω_6.

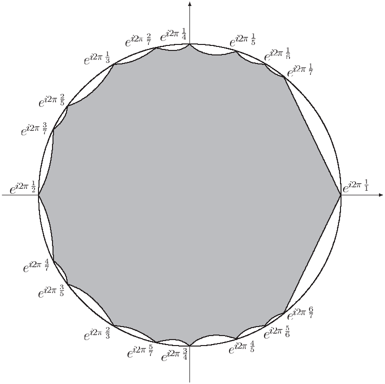


Figure: The region Ω_7.

Theorem (Romanovsky)

$$\Omega_n \cap \mathbb{T} = \{e^{i2\pi \frac{a}{b}} : 1 \leqslant a \leqslant b \leqslant n\}.$$

Proof.

Let $(a_i)_{1\leqslant i\leqslant k}$ be an arbitrary convex sequence, and let

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{pmatrix}_{n\times n} .$$

Proof.

Let $\zeta = e^{i\frac{2\pi k}{n}}$, $\overrightarrow{X} = (1, \zeta, \ldots, \zeta^{n-1})^{tr}$, and

$$\lambda = a_1 + a_2\,\zeta + \cdots + a_n\,\zeta^{n-1}.$$

Then

$$A\overrightarrow{X} = \lambda\,\overrightarrow{X}.$$

Therefore, $\lambda \in \Omega_n$, for any choice of k and any collection of convex sequence $(a_1, \ldots, a_n)$.
Indeed, we proved more! □

Theorem (Karplevich–Ito)

The region Ω_n is completely characterized as follows:

1. $\Omega_n \subset \overline{\mathbb{D}}$,
2. Ω_n *is symmetric with respect to the real axis,*
3. $\Omega_n \cap \mathbb{T} = \{e^{i2\pi\,\frac{a}{b}} : 1 \leqslant a \leqslant b \leqslant n\}$,
4. *The boundary of Ω_n consists of the above points and of some curves in $\mathbb{D}$ connecting them in circular order.*

Theorem (Karplevich–Ito)

Let $e^{i2\pi\frac{a_1}{b_1}}$ *and* $e^{i2\pi\frac{a_2}{b_2}}$ *be two consecutive points on* $\mathbb{T}$*, and suppose that* $b_1 \leq b_2$*.*
Then the equation of curve $\lambda(t)$ *connecting these two points is given by*

$$\lambda^{b_2}\left(\lambda^{b_1} - t\right)^{[n/b_1]} = (1-t)^{[n/b_1]}\,\lambda^{b_1[n/b_1]},$$

where the real parameter t *runs over the interval* $[0,1]$*.*

More about Ω_5

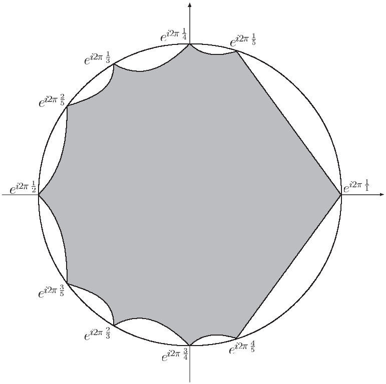


Figure: The region Ω_5.

More about Ω_5

According to Romanovski's theorem,

$$\Omega_5 \cap \mathbb{T} = \left\{ e^{i2\pi \frac{a}{b}} \;:\; \frac{a}{b} = \frac{1}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \right\}.$$

These points are ordered counterclockwise.

More about Ω_5

These points are ordered counterclockwise. According to Karplevich–Ito's theorem the boundary arcs are given by:

$$
\begin{aligned}
&\text{between } e^{i2\pi\frac{1}{1}} \text{ and } e^{i2\pi\frac{1}{5}} : && (\lambda - t)^5 = (1-t)^5,\\
&\text{between } e^{i2\pi\frac{1}{5}} \text{ and } e^{i2\pi\frac{1}{4}} : && \lambda(\lambda^4 - t) = (1-t),\\
&\text{between } e^{i2\pi\frac{1}{4}} \text{ and } e^{i2\pi\frac{1}{3}} : && \lambda(\lambda^3 - t) = (1-t),\\
&\text{between } e^{i2\pi\frac{1}{3}} \text{ and } e^{i2\pi\frac{2}{5}} : && \lambda^2(\lambda^3 - t) = (1-t),\\
&\text{between } e^{i2\pi\frac{2}{5}} \text{ and } e^{i2\pi\frac{1}{2}} : && \lambda(\lambda^2 - t)^2 = (1-t)^2.
\end{aligned}
$$

More about Ω_5

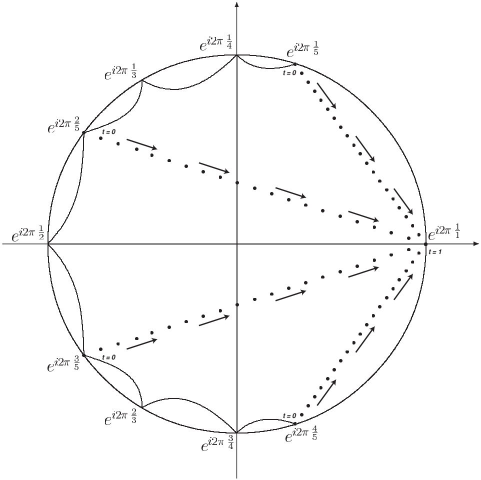
Let

$$A_{5,1} = \begin{pmatrix} t & 1-t & 0 & 0 & 0 \\ 0 & t & 1-t & 0 & 0 \\ 0 & 0 & t & 1-t & 0 \\ 0 & 0 & 0 & t & 1-t \\ 1-t & 0 & 0 & 0 & t \end{pmatrix}.$$

Then

$$\det(\lambda I - A_{5,1}) = 0 \Longrightarrow (\lambda - t)^5 = (1-t)^5.$$

More about Ω_5



More about Ω_5

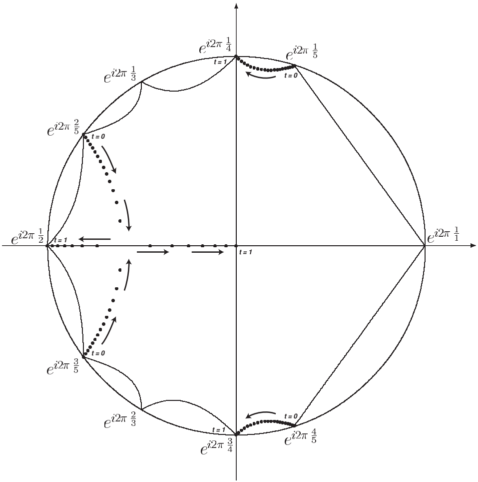
Let

$$A_{5,2} = \begin{pmatrix} 0 & 1-t & t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\det(\lambda I - A_{5,2}) = 0 \Longrightarrow \lambda\,(\lambda^4 - t) = (1-t).$$

More about Ω_5



More about Ω_5

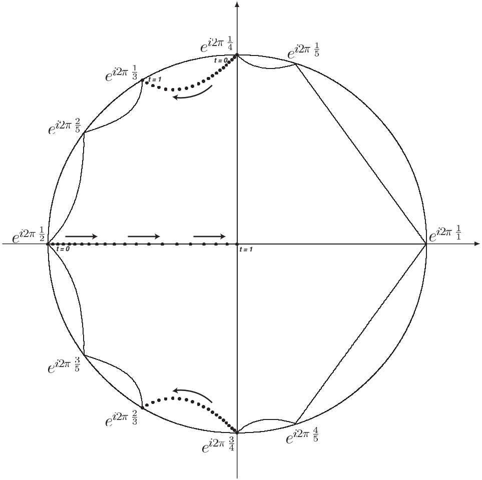
Let

$$A_{5,3} = \begin{pmatrix} 0 & 1-t & t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\det(\lambda I - A_{5,3}) = 0 \Longrightarrow (\lambda - 1)\left(\lambda(\lambda^3 - t) - (1-t)\right) = 0.$$

More about Ω_5



More about Ω_5

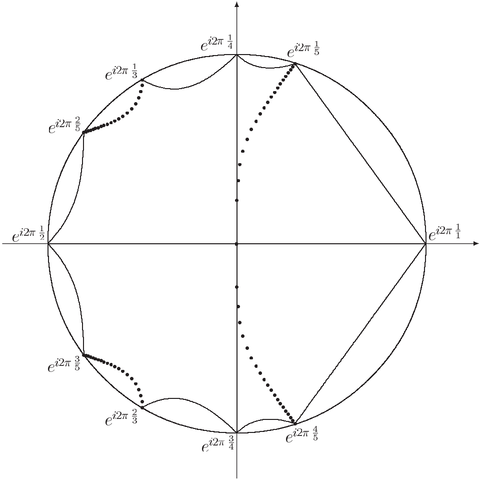
Let

$$A_{5,4} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & t & 1-t & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\det(\lambda I - A_{5,4}) = 0 \Longrightarrow \lambda^2 (\lambda^3 - t) = (1 - t).$$

More about Ω_5



More about Ω_5

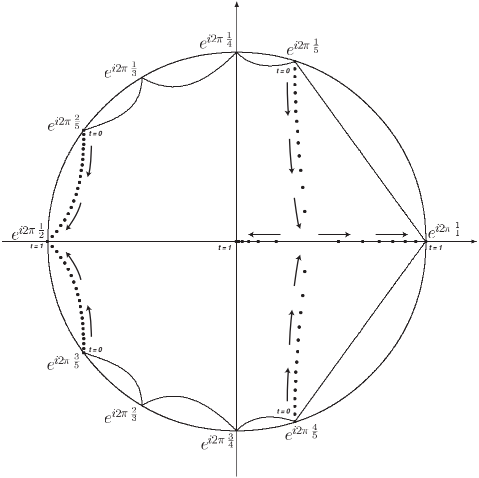
Let

$$A_{5,5} = \begin{pmatrix} 0 & 0 & 1-t & t & 0 \\ 0 & 0 & 0 & 1-t & t \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\det(\lambda I - P_{5,5}) = 0 \Longrightarrow \lambda\,(\lambda^2 - t)^2 = (1-t)^2.$$

More about Ω_5



Definition

A square matrix $A = [a_{ij}]$ is called doubly-stochastic if

1. $a_{ij} \geq 0$.
2. For each i, $\sum_{j=1}^{n} a_{ij} = 1$.
3. For each j, $\sum_{i=1}^{n} a_{ij} = 1$.

A similar question

Let

$\omega_n = \{\lambda : \lambda$ is the eigenvalues of an $n \times n$ doubly-stochastic matrix$\}$.

In other words, ω_n is the collection of *all eigenvalues* of *all $n \times n$ doubly-stochastic matrices*.

A similar question

Let

$\omega_n = \{\lambda : \lambda$ is the eigenvalues of an $n \times n$ doubly-stochastic matrix$\}$.

In other words, ω_n is the collection of *all eigenvalues* of *all $n \times n$ doubly-stochastic matrices*.

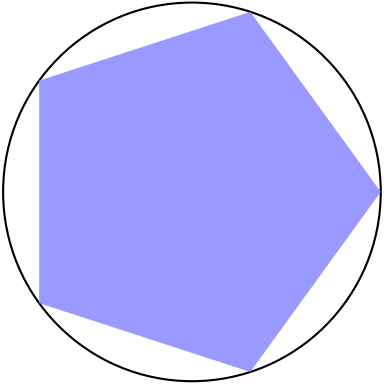
Find ω_n.

Regular n-gons

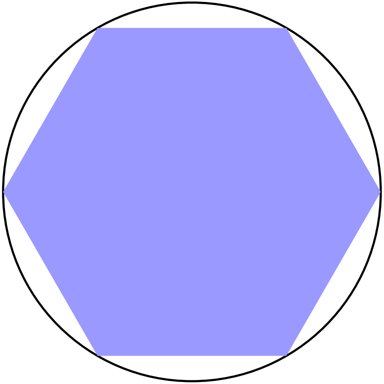
Let

$$\mathbf{P}_n := \text{The convex hull of } \{1,\ e^{i2\pi/n},\ e^{i4\pi/n},\ \ldots,\ e^{i2(n-1)\pi/n}\}.$$

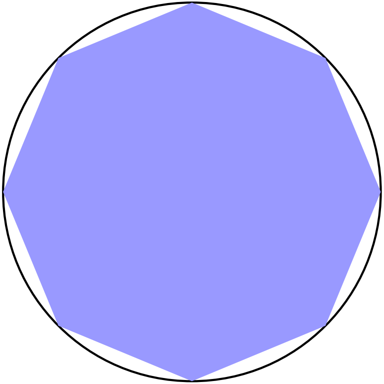
In other words, $\mathbf{P}_n$ is the closed region whose boundary is the regular n-gon with vertices at $\{1,\ e^{i2\pi/n},\ e^{i4\pi/n},\ \ldots,\ e^{i2(n-1)\pi/n}\}$.



The region $\mathbf{P}_5$



The region $\mathbf{P}_6$



The region $\mathbf{P}_8$

Inclusions

Theorem

We have

$$\bigcup_{k=1}^{n} \mathbf{P}_k \subset \omega_n \subset \Omega_n.$$

Proof.

We already saw that

$$\mathbf{P}_n \subset \omega_n.$$

Clearly $\omega_n \subset \Omega_n$.

By adding and extra row and column, we also see that

$$\omega_{n-1} \subset \omega_n.$$

□

Perfect-Mirsky conjecture (1965)

Conjecture: We have

$$\omega_n = \bigcup_{k=1}^{n} \mathbf{P}_k, \qquad (n \geq 1).$$

It is elementary to verify the conjecture for $n = 1$ and $n = 2$. Also, some easy computations approve the case $n = 3$.

Conjecture for $n = 3$ holds

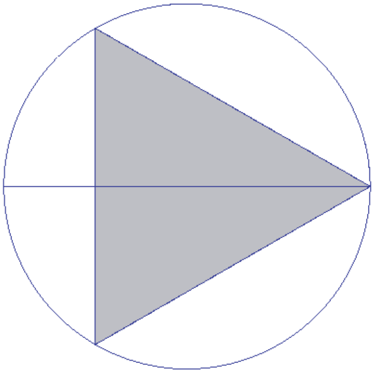


Figure: The region ω_3.

Conjecture for $n = 4$

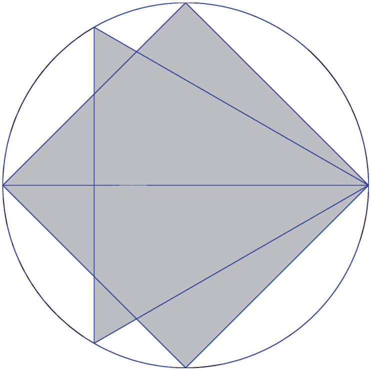


Figure: Is this ω_4?

Conjecture for $n = 5$

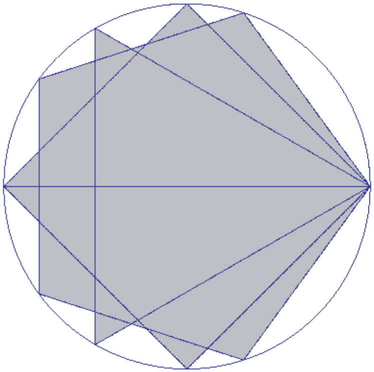


Figure: Is this ω_5?

The case $n = 5$

Theorem (JM–R. Rivard 2007)

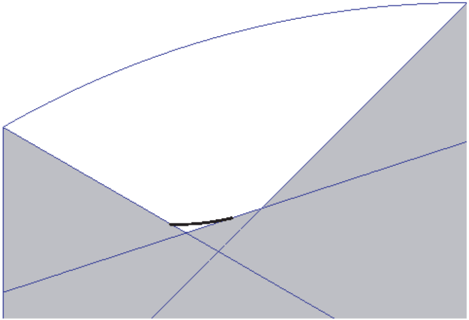
$$\bigcup_{k=1}^{5} \mathbf{P}_k \subsetneq \omega_5.$$

Counterexample ($n = 5$)

$$P_t = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & t & 0 & 1-t \\ 0 & t & 1-t & 0 & 0 \\ 0 & 1-t & 0 & 0 & t \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$t \in [0.5 - \varepsilon,\ 0.5 + \varepsilon] = [0.49, 0.51]$

Counterexample ($n = 5$)



The case $n = 4$

Theorem (Levick-Pereira-Kribs 2014)

$$\omega_4 = \bigcup_{k=1}^{4} \mathbf{P}_k.$$

Conjecture for $n = 4$ holds

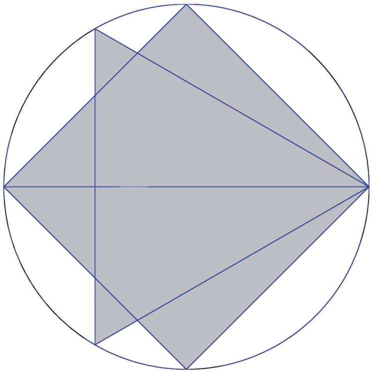


Figure: The region ω_4.

Counterexample ($n = 5$)

Theorem [JM-R. Rivard]

The matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has eigenvalues outside $\bigcup_{k=1}^{5} \mathbf{P}_k$.

Counterexample ($n = 5$)

Proof

The characteristic polynomial of A is

$$\begin{aligned}\det(\lambda I - P) &= \lambda^5 - \frac{1}{2}\lambda^4 - \frac{1}{4}\lambda^3 - \frac{1}{2}\lambda^2 + \frac{1}{4} \\ &= \frac{1}{4}(\lambda - 1)(4\lambda^4 + 2\lambda^3 + \lambda^2 - \lambda - 1).\end{aligned}$$

Put

$$f(\lambda) = 4\lambda^4 + 2\lambda^3 + \lambda^2 - \lambda - 1.$$

Counterexample ($n = 5$)

Proof

The characteristic polynomial has two other real roots and two complex roots which (using Maple) are approximately

$$a \approx 0.6449993710,$$

$$b \approx -0.5864142155,$$

and

$$\alpha \pm \beta \approx -0.2792925777 \pm 0.7635163747\, i.$$

Counterexample ($n = 5$)

Proof

Let $a, b \in \mathbb{R}$ and $\alpha \pm i\beta$ be the roots of the equation

$$4\lambda^4 + 2\lambda^3 + \lambda^2 - \lambda - 1 = 0.$$

Then, we have

$$a + b + 2\alpha = -\frac{1}{2},$$

and

$$ab(\alpha^2 + \beta^2) = -\frac{1}{4}.$$

Our goal is to show that $\alpha + i\beta \in \Delta$.

Conjecture for $n = 4$

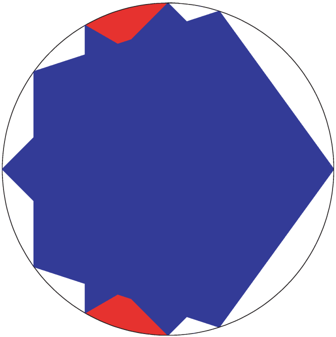


Figure: The region Δ.

Counterexample ($n = 5$)

Proof

Based on the preceding approximations, we find that

$$f(0.6449993711) > 0, \qquad \text{and} \qquad f(0.6449993709) < 0.$$

Hence, by the Intermediate Value Theorem,

$$a \in [0.6449993709, 0.6449993711].$$

Counterexample ($n = 5$)

Proof

Similarly,

$$f(-0.5864142156) > 0, \qquad \text{and} \qquad f(-0.5864142154) < 0,$$

imply that

$$b \in [-0.5864142156, -0.5864142154].$$

Counterexample ($n = 5$)

Proof

Recall that

$$a + b + 2\alpha = -\frac{1}{2}, \tag{1}$$

and

$$ab(\alpha^2 + \beta^2) = -\frac{1}{4}. \tag{2}$$

Counterexample ($n = 5$)

Proof

Therefore, by (1),

$$\alpha \in [-0.2792925778, -0.2792925776]$$

and, by (2),

$$\beta \in [0.7635163746, 0.7635163748],$$

Counterexample ($n = 5$)

Proof

In other words,

$$\begin{aligned} \alpha + i\beta \quad &\in \quad [-0.27929257785, -0.27929257765] \\ &\times \quad [0.7635163746, 0.7635163748]. \end{aligned}$$

It is now easy to verify that this rectangle is entirely in Δ.

Counterexample ($n = 5$)

Proof

Let

$$\begin{aligned} F(x,y) &= y - x - 1, \\ G(x,y) &= \sqrt{3}\,y + x - 1, \\ H(x,y) &= \frac{x - \cos(2\pi/5)}{\cos(4\pi/5) - \cos(2\pi/5)} - \frac{y - \sin(2\pi/5)}{\sin(4\pi/5) - \sin(2\pi/5)}. \end{aligned}$$

The equations

$$F = 0, \qquad G = 0, \qquad H = 0,$$

represent the three lower lines of the frontiers of Δ

Counterexample ($n = 5$)

Proof

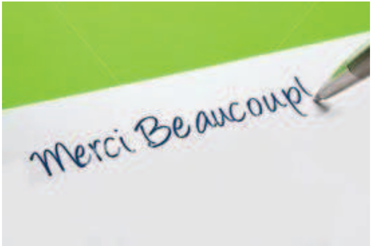
Let $(x, y) \in \mathbb{D}$, $y > 0$. Then

$$(x, y) \in \Delta$$

if and only if

$$F(x, y) > 0, \qquad G(x, y) > 0, \qquad H(x, y) > 0.$$

The above three conditions are easy to verify for the four corners of the rectangle. Done.


Merci Beaucoup!