Hadamard matrices with few distinct types

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Definition

A Hadamard matrix is a square matrix H of order n with entries ± 1 satisfying

$$HH^T = nI.$$

Example

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Sylvester matrices

$$H_0 = \begin{bmatrix} 1 \end{bmatrix}$$

$$H_{r+1} = \begin{bmatrix} H_r & H_r \\ H_r & -H_r \end{bmatrix}$$

H_r is a Hadamard matrix of order 2^r.

Equivalence

Two Hadamard matrices H_1 and H_2 are called equivalent if one is obtained from the other using some of the following operations:

- a permutation of rows
- a permutation of columns
- negations of some rows
- negations of some columns

In other words, H_1 and H_2 are equivalent if there are signed permutation matrices P and Q such that

$$H_2 = PH_1Q.$$

Classification

The number of equivalence classes of Hadamard matrices of order $n \leqslant 32$:

n	1	2	4	8	12	16	20	24	28	32
#	1	1	1	1	1	5	3	60	487	13710027

Types

$H = [h_{ij}]$: A Hadamard matrix of order n

By a sequence of row/column permutations/negations, any four distinct rows i, j, k, ℓ of H may be transformed uniquely to the form

$$\begin{array}{cccccccccc} & & s & t & t & s & t & s & s & t \\ i & : & + & + & + & + & + & + & + & + \\ j & : & + & + & + & + & - & - & - & - \\ k & : & + & + & - & - & + & + & - & - \\ \ell & : & + & - & + & - & + & - & + & - \end{array}$$

where $s + t = n/4$ and $0 \leq t \leq \lfloor n/8 \rfloor$. We define the *type* of the four rows i, j, k, ℓ as $T_{ijk\ell} = t$.

Profile of Hadamard matrices

$H = [h_{ij}]$: A Hadamard matrix of order n

For any four distinct rows i, j, k, ℓ, define

$$P_{ijk\ell} = \left| \sum_{c=1}^{n} h_{ci} h_{cj} h_{ck} h_{c\ell} \right|.$$

Let π_m be the number of four rows i, j, k, ℓ of H with $P_{ijk\ell} = m$. Then

$$\pi = (\pi_1, \pi_2, \ldots, \pi_n)$$

is called the profile of H.

Types vs. Profile

It is straightforward to check that

$$T_{ijk\ell} = \frac{n - P_{ijk\ell}}{8}.$$

Question

Is it possible to have a profile with only one nonzero entry?

in other words:

Is there a Hadamard matrix whose 4-tuples of rows are all of the same type?

The answer is almost NO!

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Lemma: Let H be a Hadamard matrix of order $n \geqslant 4$. If all 4-tuples of rows are of the same type, then $n = 4$ or $n = 12$.

Type relations

H: A Hadamard matrix of order $n = 4m$

Fix three rows and let k_i be the number of rows which are of type i with the fixed three rows. Then

$$\sum_i k_i = n - 3,$$
$$\sum_i k_i (m - 2i)^2 = m^2.$$

Also, if $k_i k_j > 0$, then

$$2(i + j) \geqslant m.$$

Next question

Is it possible to have a profile with only two nonzero entries?

in other words:

Is there a Hadamard matrix with only two distinct types for 4-tuples of rows?

Three Infinite Classes

We consider Hadamard matrices with exactly two types for 4-tuples of rows:

- Types $0, \frac{n}{8}$
- Types $1, \frac{n-4}{8}$
- Types $\frac{n}{16}, \frac{n}{8}$

Note that theses pairs of types satisfy the previous type relations.

The $(\frac{n}{16}, \frac{n}{8})$ case

Lemma: There exists no Hadamard matrix of order n whose all quadruples of rows are of type $\frac{n}{16}$ or $\frac{n}{8}$.

The Proof is not hard!
We need to consider only 7 rows.

The other two cases

Main Theorem: Let H be a Hadamard matrix of order n and $r < n/16$. Suppose that for every three distinct rows i, j, k of H, there exists a row ℓ with $T_{ijk\ell} \leqslant r$ and no row x with $r < T_{ijkx} \leqslant 2r$. Then n must be a power of 2.

Corollaries

Corollary 1: Any Hadamard matrix of order n whose all quadruples of rows are of type 0 or $\frac{n}{8}$ is equivalent to the Sylvester Hadamard matrix.

Corollary 2: Any Hadamard matrix of order n whose all quadruples of rows are of type 1 or $\frac{n-4}{8}$ has order $n = 4, 12, 20$.

Main Theorem

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Notation

Hadamard product of two $(-1,1)$-vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$:

$$a \circ b = (a_1 b_1, \ldots, a_n b_n).$$

Also let

$$\sigma(a) = |a_1 + \cdots + a_n|.$$

Then

$$\sigma(a \circ b) \geqslant \sigma(a) + \sigma(b) - n.$$

Definition: A set $\mathcal{S}$ of the rows of H is full if for every distinct rows $i, j, k \in \mathcal{S}$, the unique row ℓ with $T_{ijk\ell} \leqslant r$ is contained in $\mathcal{S}$.

Proof of Theorem

Claim: Any full set of size $s < n$ can be extended to a full set of size $2s$.

So if we start with a full set of size 4, by the above claim we find that n is a power of 2.

Proof of Claim

Let $\mathcal{S} = \{a_1, \ldots, a_s\}$ be a full set in H.

Choose an arbitrary row b_1 in H outside of $\mathcal{S}$.
Let b_i be the unique row in H such that

$$T_{a_1 a_i b_1 b_i} \leqslant r$$

for $i = 2, \ldots, s$.
It is not hard to show that

$$\mathcal{S}' = \mathcal{S} \cup \{b_1, \ldots, b_s\}$$

is a full set of size $2s$.

Problems

Classify all Hadamard matrices with only two distinct types for 4-tuples of rows.

Find another infinite family of Hadamard matrices with only two distinct types for 4-tuples of rows.