Pseudospectra and Kreiss Matrix Theorem on a General Domain

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Outline

1. Introduction
2. Pseudospectra
3. Generalization of the Kreiss Matrix Theorem

Introduction

Given a square matrix $A \in \mathbb{C}^{N\times N}$. The spectrum of A is

$$\sigma(A) := \left\{z \in \mathbb{C} : zI - A \text{ not invertible}\right\}.$$

We write $\|\cdot\|$ for the spectral norm on $\mathbb{C}^{N\times N}$, defined by $\|A\| := \sup\left\{\|Ax\|_2 : \|x\|_2 = 1\right\}$, where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{C}^N$.

We say :

- A is *power bounded* if $\sup_{n\geq 0} \|A^n\| < \infty$.
- A is *exponentially bounded* if $\sup_{t\geq 0} \|e^{tA}\| < \infty$.

Power Bounded Matrices

KMT1 : A is power bounded if and only if

$$\exists C > 0 \text{ such that } \left\|(zI - A)^{-1}\right\| \leq \frac{C}{|z| - 1} \quad (|z| > 1).$$

In particular, $\sigma(A) \subset \overline{\mathbb{D}}$ "the unit disc".
Kreiss Constant with respect to $\mathbb{D}$:

$$\mathcal{K}(\mathbb{D}) := \sup_{|z|>1} (|z| - 1)\left\|(zI - A)^{-1}\right\|.$$

Kreiss Matrix Theorem (Power matrices)

$\mathcal{K}(A) \leq \sup_{n\geq 0} \|A^n\| \leq e\, N\, \mathcal{K}(A)$,

Exponentially Bounded Matrices

KMT2 : A is exponentially bounded if and only if

$$\exists C > 0 \text{ such that } \left\|(zI - A)^{-1}\right\| \leq \frac{C}{\mathrm{Re}(z)} \quad (\mathrm{Re}(z) > 0).$$

In particular, $\sigma(A) \subset \mathcal{P}$ "the left-half plane".
Kreiss Constant with respect to $\mathcal{P}$:

$$\mathcal{K}(\mathcal{P}) := \sup_{\mathrm{Re}(z)>0} (\mathrm{Re}(z))\left\|(zI - A)^{-1}\right\|.$$

Kreiss Matrix Theorem ((Exponential matrices)

$\mathcal{K}(A) \leq \sup_{t\geq 0} \|e^{tA}\| \leq e\,N\,\mathcal{K}(A)$, The constant eN is the result of a large development following the original statement of the Kreiss matrix theorem.

- Kreiss (1962) : $\mathcal{K}(A)^{N^N}$
- Morton (1964) : $6^N(N+4)^{5N}\mathcal{K}(A)$
- Miller and Strang (1966) : $N^N\mathcal{K}(A)$
- Miller (1967) : $e^{9N^2}\mathcal{K}(A)$
- Strang and Laptev (1978) : $\frac{32}{\pi}eN^2\mathcal{K}(A)$
- Tadmor (1981) : $\frac{32}{\pi}eN\mathcal{K}(A)$
- LeVeque and Trefethen (1984) : $2eN\mathcal{K}(A)$
  Conjoncture : The optimal bound is $eN\mathcal{K}(A)$
- Smith (1985) : $(1+\frac{2}{\pi})eN\mathcal{K}(A)$
- Spijker proved the conjoncture in 1991.

Goal of the Talk

Kreiss Matrix theorem provides estimates of upper bounds of $\|A^n\|$ and $\|e^{tA}\|$ according to the resolvent norm.

Question : What about the norm $\|f(A)\|$ for an arbitrary holomorphic function f on a neighborhood of $\sigma(A)$?

Cauchy Integral Formula :

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(zI - A)^{-1} \,\mathrm{d}\, z.$$

To understand $\|f(A)\|$, it is interesting to study the resolvent norm $\left\|(zI - A)^{-1}\right\|$ of A.

Example (scholarpedia.org)

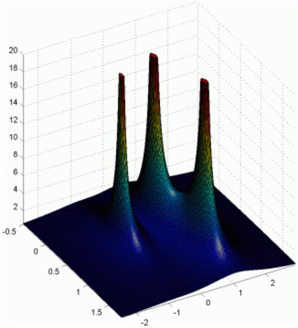
Consider the matrix

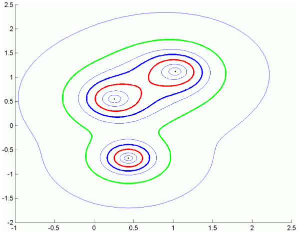
$$A = \begin{pmatrix} 1+i & 0 & i \\ -i & 0.2 & 0 \\ 0.7i & 0.2 & 0.5 \end{pmatrix}$$

The $\epsilon-$pseudospectrum of A is the set of all $z \in \mathbb{C}$ for which the graph of the function $z \mapsto \left\| (zI - A)^{-1} \right\|$ lies above the level $\frac{1}{\epsilon}$.

The boundaries of the pseudospectra of A for the values

$\epsilon = 1, 1/2, 1/3, 1/4, 1/6, 1/10, 1/20$





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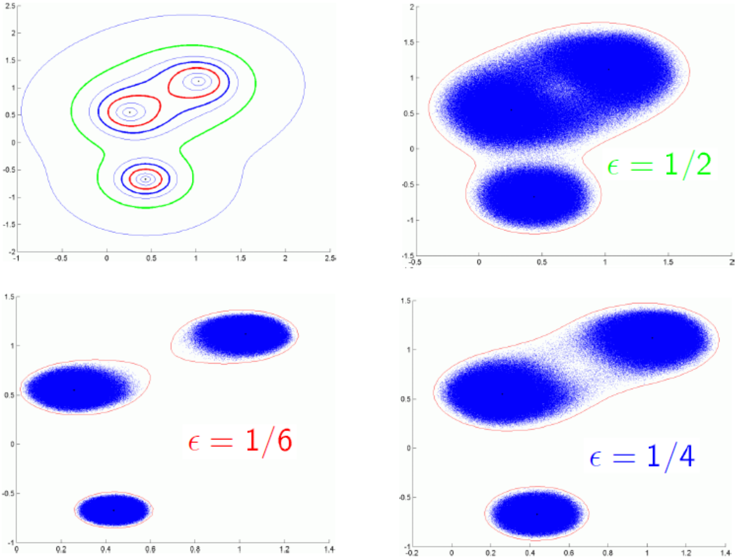
500.000 random perturbations $A + E$ with $\|E\| < \epsilon$



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Pseudospectra

The $\epsilon-$pseudospectrum of A, $\epsilon > 0$, is

$$\sigma_\epsilon(A) := \{z \in \mathbb{C} : \|(zI - A)^{-1}\| > \frac{1}{\epsilon}\}.$$

Theorem

Let $A \in \mathbb{C}^{N \times N}$ and $\epsilon > 0$ be arbitrary. TFSAE

(i) $z \in \sigma_\epsilon(A)$.

(ii) $\|(zI - A)V\| < \epsilon$ *for some* $V \in \mathbb{C}^N$ *with* $\|V\| = 1$.

(iii) $z \in \sigma(A + E)$ *for some* $E \in \mathbb{C}^{N \times N}$ *with* $\|E\| < \epsilon$.

(iv) $s_{\min}(zI - A) > \epsilon$

Identical Pseudospectra

Let A, B be $N \times N$ matrices with *identical pseudospectra*, i. e.

$$\left\|(zI - A)^{-1}\right\| = \left\|(zI - B)^{-1}\right\| \quad (\forall z \in \mathbb{C}).$$

- Must A, B be unitarily similar $(B = U^* AU)$?
- Must A, B have the same norm behavior ? i.e.

$$\left\|f(A)\right\| = \left\|f(B)\right\|$$

for all holomorphic functions f on the spectrum of A and B.

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Theorem

If A and B have identical pseudospectra, then they have the same spectrum and the same numerical range.

Theorem (Ransford-Raouafi, 2013)

Let A and B be $N \times N$ matrices with identical pseudospectra. Then, for every Möbius transformation f holomorphic on the spectrum of A and B, we have

$$\|f(A)\| \leq M \|f(B)\|,$$

where $M := \frac{5+\sqrt{33}}{2} \simeq 5,3723.$

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Theorem (Ransford-Raouafi, 2013)

Let f be a function holomorphic in a domain Ω, and suppose that f is neither constant nor a Möbius transformation. Then, given $N \geq 6$ and $M > 1$, there exist $N \times N$ matrices A, B with spectra in Ω, such that

$$\left\|(zI - A)^{-1}\right\| = \left\|(zI - B)^{-1}\right\| \quad (\forall z \in \mathbb{C})$$

and

$$\left\|f(A)\right\| > M\left\|f(B)\right\|.$$

Super-identical pseudospectra

Two matrices $A, B \in \mathbb{C}^{N\times N}$ have *super-identical pseudospectra* if

$$s_k(zI - A) = s_k(zI - B) \quad (\forall z \in \mathbb{C} \text{ and } k = 1, \ldots, N).$$

Theorem (Ransford, 2007)

If $A, B \in \mathbb{C}^{N\times N}$ have super-identical pseudospectra, then, for any function f holomorphic on their spectrum,

$$\frac{1}{\sqrt{N}} \le \frac{\|f(A)\|}{\|f(B)\|} \le \sqrt{N}.$$

Theorem (Armentia–Gracia–Velasco, 2012)

If A, B have super-identical pseudospectra, then they are similar.

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Theorem (Armentia–Gracia–Velasco, 2012)

If A, B have super-identical pseudospectra, then they are similar.

Theorem (D. Farenick et al., 2011)

Let A be an upper triangular Toeplitz matrix with nonzero superdiagonal, and let B be any matrix of the same size. Then A and B are unitarily similar if and only if they have super-identical pseudospectra.

Theorem (D. Farenick et al., 2011)

Let A and B be an $N \times N$ upper triangular matrices that are indecomposable with respect to similarity. Then A and B are unitarily similar if and only if A_k and B_k have super-identical pseudospectra for all $k = 1, \cdots, N$, where A_k and B_k are the leading principal $k \times k$ submatrices of A and B respectively.

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Holomorphic Functions on the Unit Disc

Denote by $\mathcal{A}(\overline{\mathbb{D}})$ the set of holomorphic functions on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$.

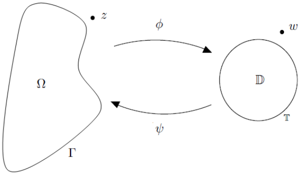
Theorem (Vitse, 2005)

Suppose A is an $N \times N$ matrix such that $\sigma(A) \subset \mathbb{D}$ and $\mathcal{K}(\mathbb{D}) < \infty$. Then, for all $f \in \mathcal{A}(\overline{\mathbb{D}})$,

$$\|f(A)\| \leq \frac{16}{\pi} \mathcal{K}(\mathbb{D}) \, N \|f\|_{\mathbb{D}},$$

where $\|f\|_{\mathbb{D}} := \max_{|z|=1} |f(z)|$.

General Complex Domain



- Riemann Mapping Theorem : there is a unique conformal map ϕ from Ω^c to $\mathbb{D}^c$ normalized by $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$,

$$w = \phi(z) := dz + d_0 + \sum_{k=1}^{\infty} \frac{d_k}{z^k}, \quad (d > 0), \quad z \in \Omega^c.$$

- Kreiss Constant with respect to Ω :

$$\mathcal{K}(\Omega) := \sup_{z \notin \Omega} \frac{|\phi(z)| - 1}{|\phi'(z)|} \left\| (zI - A)^{-1} \right\|.$$

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Arbitrary disc in the complex plane

Theorem

Let D be an arbitrary disc on the complex plane. Suppose A is an $N \times N$ matrix with $\sigma(A) \subset D$ and $\mathcal{K}(D) < \infty$. Then, for all function $f \in \mathcal{A}(D)$,

$$\|f(A)\| \leq \frac{16}{\pi} \mathcal{K}(D)\, N \|f\|_D,$$

where $\|f\|_D := \max_{z \in D} |f(z)|$.

What about general complex domains?

- The n^{th} *Faber polynomial* $F_n(z)$, for $n = 0, 1, 2, \cdots$, associated with Ω is the polynomial part of $[\phi(z)]^n$.
- $F_n(z)$ is a polynomial of degree n.

Theorem (Toh-Trefethen, 1999)

Let Ω be a compact subset of the complex plane such that its complementary Ω^c is simply connected in the extended complex plane. Suppose A is an $N \times N$ complex matrix with $\sigma(A) \subset \Omega$ and $\mathcal{K}(\Omega) < \infty$. If the boundary of Ω is twice continuously differentiable, then for all $n \geq 0$,

$$\|F_n(A)\| \leq C_\Omega \, e \, N \, \mathcal{K}(\Omega),$$

where the constant C_Ω depends only on Ω.

Note that if Ω is the unit disk, we have $F_n(A) = A^n$.

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Note that if Ω is the unit disk, we have $F_n(A) = A^n$.

Theorem (Toh-Trefethen, 1999)

Let Ω be a compact subset of the complex plane such that its complementary Ω^c is simply connected in the extended complex plane. Suppose A is a bounded linear operator in Hilbert space, with $\sigma(A) \subset \Omega$ and $\mathcal{K}(\Omega) < \infty$. Then for all $n \geq 0$,

$$\|F_n(A)\| \leq e\,(n+1)\,\mathcal{K}(\Omega), \tag{1}$$

Converely, if $\sup_{n\geq 0} \|F_n(A)\| < \infty$, then $\sigma(A) \subset \Omega$, $\mathcal{K}(\Omega)$ is finite, and

$$\mathcal{K}(\Omega) \leq \sup_{n\geq 0} \|F_n(A)\|. \tag{2}$$

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A Markov function is a function of the form

$$f(z) := \int_{\alpha}^{\beta} \frac{d\mu(x)}{z-x}, \qquad (3)$$

where μ is a positive measure with $supp(\mu) \subset [\alpha, \beta]$ for $-\infty \leq \alpha < \beta < \infty$.

Example :

- $\frac{\log(1+z)}{z} = \int_{-\infty}^{-1} \frac{\frac{-1}{x}\,\mathrm{d}\,x}{z-x} \quad (z \notin (-\infty, -1])$
- $z^{\gamma} = \int_{-\infty}^{0} \frac{-|x|^{\gamma}\,\mathrm{d}\,x}{z-x}, \qquad -1 < \gamma < 0 \quad (z \notin (-\infty, 0]).$

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Theorem

Let Ω be a symmetric compact subset of the complex plane such that its complementary Ω^c is simply connected in the extended complex plane. Suppose A is a linear bounded operator in a Hilbert space, with $\sigma(A) \subset \Omega$ and $\mathcal{K}(\Omega) < \infty$. If f is a Markov function defined by (3), then

$$\|f(A)\| \leq \mathrm{e}\, C_{\Omega}^{\alpha,\beta}\, \mathcal{K}(\Omega)\, \|f\|_{\Omega},$$

where the constant $C_{\Omega}^{\alpha,\beta}$ depends only on Ω, α and β.

Example 1 : Let $\Omega = \overline{\mathbb{D}}$ the closed unit disc. Suppose A is a linear bounded operator in a Hilbert space, with $\sigma(A) \subset \overline{\mathbb{D}}$ and $\mathcal{K}(\overline{\mathbb{D}}) < \infty$. If f is a Markov function defined by (3), with $\beta < -1$, then

$$\|f(A)\| \leq \mathrm{e}\,\frac{\beta^2}{(1+\beta)^2}\,\mathcal{K}(\mathbb{D})\,\|f\|_{\mathbb{D}},$$

Example 2 : Let Ω the closed ellipse with foci at ± 1 and semi-axes $a = \frac{1}{2}(R + \frac{1}{R})$ and $b = \frac{1}{2}(R - \frac{1}{R})$ for some $R > 1$. Suppose A is a linear bounded operator in a Hilbert space, with $\sigma(A) \subset \Omega$ and $\mathcal{K}(\Omega) < \infty$. If f is a Markov function defined by (3), with $\beta < -a$, then

$$\|f(A)\| \leq \mathrm{e}\,\frac{(\sqrt{\beta^2 - 1} - \beta)^2}{(\sqrt{\beta^2 - 1} - \beta - R)^2}\,\mathcal{K}(\Omega)\,\|f\|_{\Omega},$$

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Thank you for your attention !