Bezout equations for stable rational matrix functions: the least squares solution and description of all solutions

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Dedicated to Peter Lancaster, a wonderful mathematician and a great friend.

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Problem

By $RH^{\infty}_{p\times q}$ we denote all stable rational $p \times q$ matrix functions. Here *stable* means all poles are outside the closed unit disc. Such functions are analytic on the open unit disc $\mathbb{D}$ and continuous on the closed unite disc $\overline{\mathbb{D}}$. Hence they are matrix-valued H^{∞} functions as well as H^2 functions.

Problem. Given $G \in RH^{\infty}_{p\times q}$, $p \leq q$, find $X \in RH_{q\times p}$ such that

$$G(z)X(z) = I_p \qquad [I_p \text{ is the } p \times p \text{ identity matrix}]$$

Example. $G(z) = \begin{bmatrix} 1+z & -z \end{bmatrix}$. Thus $p = 1$ and $q = 2$. We have

$$G(z)X(z) = 1 \iff (1+z)x_1(z) - zx_2(z) = 1 \quad [\text{classical Bezout}]$$

$$X(z) \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies G(z)X(z) = 1.$$

Main aims

We are interested in

(a) conditions of existence of solutions

(b) least squares solution

(c) description of all solutions

Existence of solutions

With $G \in RH^\infty_{p\times q}$ we associate the analytic Toeplitz operator T_G given by:

$$T_G = \begin{bmatrix} G_0 & & & \\ G_1 & G_0 & & \\ G_2 & G_1 & G_0 & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} : \ell^2_+(\mathbb{C}^q) \to \ell^2_+(\mathbb{C}^p).$$

$$\ell^2_+(\mathbb{C}^k) \equiv H^2(\mathbb{C}^k) \quad \Longrightarrow \quad T_G \equiv M_G$$

It follows that

$$G(z)X(z) = I_{p\times p} \quad (z \in \mathbb{D}) \Rightarrow T_G T_X = T_{GX} = I_{\ell^2_+(\mathbb{C}^m)}$$
$$\Rightarrow T_G \text{ right invertible.}$$

THM. *Let* $G \in RH^\infty_{p\times q}$*. Then the equation*

$$G(z)X(z) = I_p \qquad (\star)$$

has a solution $X \in RH^\infty_{q\times p}$ *if and only if the Toeplitz operator* T_G *is right invertible. Moreover, in that case* $T_G T_G^*$ *is invertible and the function*

$$X(\cdot) := \mathcal{F}_{\mathbb{C}^p}\Big(T_G^*(T_G T_G^*)^{-1}E_p\Big), \ \text{where} \ E_p := \begin{bmatrix} I_p \\ 0 \\ 0 \\ \vdots \end{bmatrix},$$

is in $RH^\infty_{p\times q}$ *and satisfies the Bezout equation* $(\star)$*. Furthermore,* X *is the least squares solution, that is, for any other solution* $Y \in RH^\infty_{q\times p}$ *we have*

$$\|T_X E_p u\|_{\ell^2_+(\mathbb{C}^q)} \leq \|T_Y E_p u\|_{\ell^2_+(\mathbb{C}^q)} \ \text{for each } u \text{ in } \mathbb{C}^p.$$

N.B. The operator $T_G^*(T_G T_G^*)^{-1}$ is the Moore-Penrose inverse of T_G.

Computing solutions by using state space methods (1)

$G \in RH^{\infty}_{p\times q}$ admits a finite dimensional state space realization, that is, G can be written as:

$G(z) = D + zC(I_n - zA)^{-1}B$, where

A, B, C, D are matrices of appropriate sizes, and

A is stable, that is all eigenvalues of A are in the open unit disc $\mathbb{D}$.

Given the realization of G we let P be the unique solution of the Stein equation $P - APA^* = BB^*$, that is, $P = \sum_{n=0}^{\infty} A^n BB^* A^{*n}$. Furthermore, we consider the algebraic Riccati equation:

$$\text{(ARE)} \qquad Q = A^*QA + (C - \Lambda^*QA)^*(R_0 - \Lambda^*Q\Lambda)^{-1}(C - \Lambda^*QA)$$

$$\text{where } R_0 = DD^* + CPC^* \text{ and } \Lambda = BD^* + APC^*.$$

Computing solutions by using state space methods (2)

$$\text{(ARE)}\ Q = A^*QA + (C - \Lambda^*QA)^*(R_0 - \Lambda^*Q\Lambda)^{-1}(C - \Lambda^*QA)$$

$$P - APA^* = BB^*$$

THM. *The operator T_G is right invertible if and only if*

(1) *the ARE has a (unique) stabilizing solution Q, that is,*

> (a) *Q is an $n \times n$ matrix satisfying* (ARE),
>
> (b) *$R_0 - \Lambda^*Q\Lambda$ is positive definite,*
>
> (c) *the matrix $A_0 := A - \Lambda(R_0 - \Lambda^*Q\Lambda)^{-1}(C - \Lambda^*QA)$ is stable.*

(2) *the matrix $I_n - PQ$ is non-singular.*

Computing solutions by using state space methods (3)

(ARE) $Q = A^* QA + (C - \Lambda^* QA)^* (R_0 - \Lambda^* Q\Lambda)^{-1}(C - \Lambda^* QA)$

$$P - APA^* = BB^*$$

THM 1. *Assume the ARE has a stabilizing solution Q and $I_n - PQ$ is non-singular. Then the least squares solution Φ is given by*

$$\Phi(z) = \Big(I_p - zC_1(I_n - zA_0)^{-1}(I_n - PQ)^{-1}B\Big)D_1,$$

where

$$A_0 = A - \Lambda(R_0 - \Lambda^* Q\Lambda)^{-1}(C - \Lambda^* QA), \quad [A_0 \textit{ is stable}]$$

$$C_1 = D^* C_0 + B^* QA_0,$$

$$\textit{with } C_0 := (R_0 - \Lambda^* Q\Lambda)^{-1}(C - \Lambda^* QA),$$

$$D_1 = (D^* - B^* Q\Lambda)(R_0 - \Lambda^* Q\Lambda)^{-1} + C_1(I_n - PQ)^{-1}PC_0^*.$$

Computing solutions by using state space methods (4)

THM 2. *Assume the ARE has a stabilizing solution Q and $I_n - PQ$ is non-singular. Then all solutions are given by $X = \Phi + \Theta F$. Here Φ is the least squares solution, the free parameter F is an arbitrary function in $RH^\infty_{(q-p)\times p}$ and $\Theta \in RH^\infty_{q\times(q-p)}$ is given by*

$$\Theta(z) = \Big(I_q - zC_1(I_n - zA_0)^{-1}(I_n - PQ)^{-1}B\Big)\hat{D}.$$

Here A_0 and C_1 are as on the previous slide, and $\hat{D}$ is any one-to-one $q \times (q-p)$ matrix such that

$$\hat{D}\hat{D}^* = I_q - (D^* - B^*Q\Lambda)(R_0 - \Lambda^*Q\Lambda)^{-1}(D - \Lambda^*QB) + \\ - B^*QB - C_1(I_n - PQ)^{-1}PC_1^*.$$

Furthermore, $\hat{D}$ is uniquely determined up to a constant unitary matrix on the right and Θ is inner.

Back to the example $G(z) = \begin{bmatrix} 1+z & -z \end{bmatrix}$

$$G(z)X(z) = 1 \iff (1+z)x_1(z) - zx_2(z) = 1 \quad \text{[classical Bezout]}$$

We already know that $X(z) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is solution. Questions: what is the least square solution, all solutions?

We apply our theorems. A stable realization of G is given by

$$A = 0, \quad B = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad C = 1, \quad D = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The solution P of the Stein equation $P - APA^* = BB^*$ is given by $P = 2$,

$$R_0 = DD^* + CPC^* = 3 \quad \text{and} \quad \Lambda = BD^* + APC^* = 1.$$

The corresponding ARE is $Q = (3 - Q)^{-1}$.

Example – cont'd

The corresponding ARE is $Q = (3 - Q)^{-1}$, which has two solutions: $Q = \frac{1}{2}(3 \pm \sqrt{5})$. The stabilizing solution is given by $Q = \frac{1}{2}(3 - \sqrt{5})$. Indeed, for this Q we have

$$R_0 - \Lambda^* Q \Lambda = 3 - Q = \frac{1}{2}(3 + \sqrt{5}) > 0;$$

$$A_0 = A - \Lambda(R_0 - \Lambda^* Q \Lambda)^{-1}(C - \Lambda^* Q A) = Q, \text{ and thus } A_0 \text{ is stable.}$$

Furthermore, $I - PQ = \sqrt{5} - 2 \neq 0$.

Then **THM 1** shows that for $G(z) = \begin{bmatrix} 1 + z & -z \end{bmatrix}$ the least squares solution of $G(z)X(z) = 1$ is given

$$X(z) = \frac{Q}{1 - 2Q}(1 + zQ)^{-1} \begin{bmatrix} 1 - Q \\ Q \end{bmatrix}, \text{ where } Q = \tfrac{1}{2}(3 - \sqrt{5}).$$

Example – cont'd

Furthermore, **THM 2** shows that for $G(z) = \begin{bmatrix} 1+z & -z \end{bmatrix}$ all stable rational 2×1 matrix solutions Y of $G(z)Y(z) = 1$ are given by

$$Y(z) = X(z) + \Theta(z)\varphi(z),$$

where φ is any scalar stable rational function and

$$\Theta(z) = \sqrt{Q}(1 + zQ)^{-1} \begin{bmatrix} z \\ 1+z \end{bmatrix}, \text{ with } Q = \tfrac{1}{2}(3 - \sqrt{5}).$$

Where does the ARE come from?

Put $R(z) = G(z)G(\bar{z}^{-1})^*$. Let $\{R_j\}_{j\in\mathbb{Z}}$ be the Fourier coefficients of R.

$$T_R := \begin{bmatrix} R_0 & R_{-1} & R_{-2} & \cdots \\ R_1 & R_0 & R_{-1} & \cdots \\ R_2 & R_1 & R_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} : \ell^2_+(\mathbb{C}^p) \to \ell^2_+(\mathbb{C}^p). \quad [T_R \neq T_G T_G^*]$$

$$R(z) = zC(I - zA)^{-1}\Lambda + (DD^* + CPC^*) + \Lambda^*(zI - A^*)^{-1}C^* \quad (z \in \mathbb{T})$$

THM. *The operator T_R is invertible if and only if*

$$Q = A^*QA + (C - \Lambda^*QA)^*(R_0 - \Lambda^*Q\Lambda)^{-1}(C - \Lambda^*QA)$$

has a stabilizing solution Q. Moreover in that case $Q := W_{obs}^* T_R^{-1} W_{obs}$, where $W_{obs} = \mathrm{col}\,[CA^j]_{j=0}^{\infty}$.

Thank you for your attention!