# Skew-symmetric EW matrices and tournaments

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# Introduction

- Let $X$ be a skew-symmetric $n \times n$ $(1,-1)$-matrix: $X + X^\top = 2I$.
- Normalize $X$ so that the first row consists of $1$ and the first column consists of $-1$ except for the $(1,1)$-entry:

$$X = \begin{pmatrix} 1 & \mathbf{1}^\top \\ -\mathbf{1} & I + A - A^\top \end{pmatrix}$$

for some $(0,1)$-matrix $A$ such that $A + A^\top = J - I$.

When can we characterize the skew-symmetric $(1,-1)$-matrix $X$ in terms of the $(0,1)$-matrix $A$?

- Known results: Skew-symmetric Hadamard matrices and tournaments.
- Today's topic: Skew-symmetric EW matrices and tournaments.

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# Hadamard matrices

- For an $n \times n$ $(1, -1)$-matrix $X$,

$$|\det(X)| \leq n^{n/2}.$$

  Equality holds if and only if $X$ is a Hadamard matrix, that is $XX^\top = nI$.
- The order of a Hadamard matrix must be $1$, $2$, or a multiple of $4$.
- Hadamard conjecture: Hadamard matrices exist for all such orders.
- It is also conjectured that skew-symmetric Hadamard matrices exist for all order divisible by $4$.

# Equivalences of skew-symmetric Hadamard matrices

- A *tournament* is a digraph whose adjacency matrix $A$ satisfies that $A + A^\top = J - I$.
- A tournament of order $4t+3$ is *doubly regular* if $AA^\top = (t+1)I + tJ$.

**Theorem**

The existence of the following are equivalent:

(1) A skew-symmetric Hadamard matrix of order $n$.

(2) A doubly regular tournament of order $n-1$. (Reid-Brown, 1972)

(3) A tournament of order $n-1$ with spectrum $\{(\frac{n}{2}-1)^1, (\frac{-1\pm\sqrt{-n}}{2})^{n/2-1}\}$. (Zagaglia Salvi, 1984)

(4) A regular tournament of order $n-1$ with three distinct eigenvalues. (Rowlinson, 1986)

(5) An irreducible tournament of order $n$ having 4 distinct eigenvalues, one of which is zero with algebraic multiplicity 1. (Kirkland-Shader, 1994)

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# Equivalences of skew-symmetric Hadamard matrices

| | skew-symmetric $(1,-1)$-matrix | $(0,1)$-matrix (combinatorial) | $(0,1)$-matrix (spectrum) |
|---|---|---|---|
| order $n$ | $XX^\top = nI$ | | four distinct eigenvalues |
| order $n-1$ | $XX^\top = nI - J$ | doubly regular tournament | three distinct eigenvalues |

# Maximal det. of $(\pm 1)$-matrices of order $n \equiv 2 \pmod 4$

- Ehlich (1964) and Wojtas (1964) independently showed that for an $n \times n$ $(1,-1)$-matrix $X$ where $n \equiv 2 \pmod 4$,

$$|\det(X)| \leq 2(n-1)(n-2)^{(n-2)/2}.$$

Equality holds if and only if there exists an $n \times n$ $(1,-1)$-matrix $B$ such that

$$BB^\top = B^\top B = \begin{pmatrix} (n-2)I_{n/2} + 2J_{n/2} & O_{n/2} \\ O_{n/2} & (n-2)I_{n/2} + 2J_{n/2} \end{pmatrix} \quad (1)$$

- A $(1,-1)$-matrix $B$ is an *EW matrix* if the equation (1) holds.

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# Results of EW matrices

- An EW matrix of order $n$ exists only if $2n-2$ is a sum of two squares.
- There exists an EW matrix of order $2(q^2+q+1)$ for any prime power $q$ (Koukouvinos, Kounias, Sebbery, 1991).
- Armario and Flau (2016) showed that if there exists a skew-symmetric EW matrix of order $n$, then $2n-3$ must be square.
- Skew-symmetric EW matrices exist for order $n = 6, 14, 26, 42, 62$.

**Problem**

Do there exist infinitely many of skew-symmetric EW matrices?

# Skew-symmetric EW matrices and tournaments

**Theorem(Armario, 2015, *ADTHM*)**

The existence of the following are equivalent:

(1) A skew-symmetric EW matrix of order $4t+2$.

(2) A tournament of order $4t+1$ satisfying

$$AA^\top = t(I_{4t+1} + J_{4t+1}) + \begin{pmatrix} -J_t & -J_t & -J_{t,a} & -J_{t,2t+1-a} \\ -J_t & J_t & 0 & 0 \\ -J_{a,t} & 0 & 0 & -J_{a,2t+1-a} \\ -J_{2t+1-a,t} & 0 & -J_{2t+1-a,a} & 0 \end{pmatrix}$$

for some $a$.

**Problem(Armario, 2015)**

Characterize the above tournament of order $4t+1$ by its spectrum.

# Main theorem

## Theorem(Greaves-S.)

The existence of the following are equivalent:

(1) A skew-symmetric EW matrix of order $4t+2$.

(2) A tournament of order $4t+1$ with characteristic polynomial

$$\chi(t) = (x^3 - (2t-1)x^2 - t(4t-1))(x^2 + x + t)^{2t-1}.$$

- In proof, we make use of main angles of $S := X - I$ or $A - A^\top$, where $X$ is a skew-symmetric $(1,-1)$-matrix and $A$ is a tournament matrix.

| | skew-symmetric $(1,-1)$-matrix | $(0,1)$-matrix (combinatorial) | $(0,1)$-matrix (spectrum) |
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| order $n$ | EW matrices | | |
| order $n-1$ | | Armario | Greaves-S. |

# Main angle

- The main angle was introduced by Cvetković in $1972$ for simple undirected graphs.
- The notion of the main angle can be extended to normal matrices:
  - $M$: a normal matrix, i.e. $MM^* = M^*M$.
  - $M = \sum_{i=1}^{s} \tau_i P_i$: the spectral decomposition.
  - Define the main angle $\alpha_i$ by

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# The characteristic polynomial of Seidel and adjacency matrices of tournaments

- $A$: the adjacency matrix of a tournament.
- $S = A - A^\top = 2A - J + I$: the Seidel matrix.
- $\tau_i, \alpha_i$ $(i = 1, 2, \ldots, s)$: the eigenvalue and the corresponding main angle of $S$.
- $\chi_M(t) := \det(M - tI)$: the characteristic polynomial of $M$.

**Lemma**

$\chi_A(x) = (\frac{-1}{2})^n \chi_S(-2x-1)(1 + \sum_{i=1}^s \frac{\alpha_i}{\tau_i - (2x+1)})$.

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# Proof: (1)⇒(2)

(1) There exists a skew-symmetric EW matrices of order $4t+2$.

(2) There exists a tournament of order $4t+1$ with characteristic polynomial $\chi(t) = (x^3-(2t-1)x^2-t(4t-1))(x^2+x+t)^{2t-1}$.

[Proof of (1)⇒(2)]

- Let $S+I$ be a normalized skew-symmetric EW matrix of order $4t+2$.
- Determine the spectrum of $S$: $\lambda = \sqrt{-8t-1}, \mu = \sqrt{-4t+1}$. spec$(S) = \{[\pm\lambda]^1, [\pm\mu]^{2t}\}$ and $\alpha_{\pm\lambda} = \frac{4t+1}{2t+1}, \alpha_{\pm\mu} = \frac{2t}{2t+1}$.
- The characteristic polynomial of $A$ is uniquely determined from the data of $S$: $\chi_A(x) = x(x^3-(2t-1)x^t(4t-1))(x^2+x+t)^{2t-1}$.
- Let $S = A - A^\top$. Then $A$ has the form $A = \begin{pmatrix} 0 & \mathbf{0}^\top \\ \mathbf{1} & A' \end{pmatrix}$, so $A'$ is a tournament and its char. poly. is the desired form.

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# Proof: (2)⇒(1)

[Proof of (2)⇒(1)] Let $A$ be a tournament matrix with $\chi_A(x) = (x^3 - (2t-1)x^2 - t(4t-1))(x^2 + x + t)^{2t-1}$.

**Lemma**

Let $A$ be a tournament matrix and let $\theta$ be an eigenvalue of $A$ with multiplicity $m$. If $\mathrm{Re}(\theta) = -\frac{1}{2}$, then $-2\mathrm{Im}(\theta)$ is an eigenvalue of $S$ with multiplicity at least $m$.

- Since $A$ has eigenvalues $\frac{-1\pm\sqrt{1-4t}}{2}$ with multiplicity $2t-1$, $S$ has eigenvalues $\mu = \mp\sqrt{1-4t}$ with multiplicity $\geq 2t-1$.
- Determine the remaining three eigenvalues of $S$: $0$, $\lambda = \pm\sqrt{1-8t}$.
- Determine the main angles: $\alpha_\lambda = 0$, $\alpha_0 = \frac{(8t+1)(4t-1)}{8t-1}$, $\alpha_\mu = \frac{4t}{8t-1}$.
- Then $S_1 := \begin{pmatrix} 0 & \mathbf{1}^\top \\ -\mathbf{1} & S \end{pmatrix}$ has characteristic polynomial $\chi_{S_1}(x) = (x^2 + 8t + 1)(x^2 + 4t - 1)^{2t}$.
- It remains to show that $S_1 S_1^\top - (4t-1)I$ is signed permutation equivalent to $\begin{pmatrix} 2J_{2t+1} & O \\ O & 2J_{2t+1} \end{pmatrix}$.

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Let $A$ be a tournament matrix and let $\theta$ be an eigenvalue of $A$ with multiplicity $m$. If $\mathrm{Re}(\theta) = -\frac{1}{2}$, then $-2\mathrm{Im}(\theta)$ is an eigenvalue of $S$ with multiplicity at least $m$.

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Let $S = X - I$.

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(1) implies (2).

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Does (2) imply (1)?

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