Reducing Matrix Polynomials to Simpler Forms

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[1] Joint work with Yuji Nakatsukasa, Leo Taslaman and Françoise Tisseur.

Simple Forms

- Diagonal form:

$$D(\lambda) = d_1(\lambda) \oplus d_2(\lambda) \oplus \cdots \oplus d_n(\lambda)$$

- Triangular form:

$$T(\lambda) = \begin{bmatrix} t_{11}(\lambda) & t_{12}(\lambda) & \dots & t_{1n}(\lambda) \\ & t_{22}(\lambda) & & t_{2n}(\lambda) \\ & & \ddots & \vdots \\ & & & t_{nn}(\lambda) \end{bmatrix}$$

- Hessenberg form:

$$H(\lambda) = \begin{bmatrix} h_{11}(\lambda) & h_{12}(\lambda) & h_{13}(\lambda) & \cdots & h_{1n}(\lambda) \\ h_{21}(\lambda) & h_{22}(\lambda) & h_{23}(\lambda) & \cdots & h_{2n}(\lambda) \\ & h_{32}(\lambda) & h_{33}(\lambda) & \cdots & h_{3n}(\lambda) \\ & & \ddots & \ddots & \vdots \\ & & & h_{nn-1}(\lambda) & h_{nn}(\lambda) \end{bmatrix}$$

Preliminaries

$P(\lambda) = I_n\lambda^\ell + P_{\ell-1}\lambda^{\ell-1} + \ldots + P_1\lambda + P_0 \in \mathbb{F}[\lambda]^{n\times n}$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Observations

- The simpler form must preserve the size, the degree and the elementary divisors.
- Reduction to triangular form by *strict equivalence*, $EP(\lambda)F = T(\lambda)$, is possible only in very special cases.
- Reduction to triangular form by *unimodular transformations*, $U(\lambda)P(\lambda)V(\lambda) = T(\lambda), \det U(\lambda), \det V(\lambda)$ non-zero constant, is always possible[2] if $\mathbb{F} = \mathbb{C}$ but its numerical implementation is not trivial.
- If $\mathbb{F} = \mathbb{R}$, by *unimodular transformations*, $P(\lambda)$ can be reduced to **block**-triangular form with 2×2 and 1×1 blocks in the diagonal.

$$\begin{bmatrix} \lambda^2-\lambda & -(\lambda-1) \\ \lambda-1 & \lambda^2-\lambda \end{bmatrix} \quad (\lambda-1) \mid (\lambda-1)(\lambda^2+1)$$ [3]

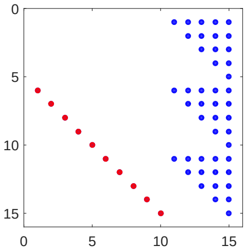
[2] L. Taslaman, F. Tisseur , I. Z.: Triangularizing matrix polynomials, LAA, 2013
[3] F. Tisseur , I. Z.: Triangularizing quadratic matrix polynomials, SIMAX, 2013

Triangular form

```
n = 5; deg = 3; % size and degree
% coefficient matrices
P0 = randn(n); P1 = randn(n); P2 = randn(n);
C_P = [ zeros(n) zeros(n) -P0; % left companion form
       eye(n) zeros(n) -P1;%
      zeros(n) eye(n) -P2 ];%
```

$$C_L(P) = \begin{bmatrix} 0 & 0 & -P_0 \\ I & 0 & -P_1 \\ 0 & I & -P_2 \end{bmatrix}$$

```
[U,~] = schur(C_P,'complex');
X = U*kron(eye(n), ones(deg,1));
S = [X  C_P*X  C_P^(deg-1)*X];
C_R = S\C_P*S;
spy(abs(C_R)>1e-12)
```

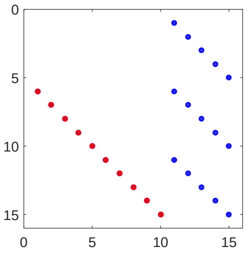


Diagonal form

```
n = 5; deg = 3; % size and degree
P0 = randn(n); P1 = randn(n); P2 = randn(n); %
coefficient matrices
C_P = [ zeros(n) zeros(n) -P0; % left companion form
      eye(n) zeros(n) -P1;
     zeros(n) eye(n) -P2 ];
```

$$C_L(P) = \begin{bmatrix} 0 & 0 & -P_0 \\ I & 0 & -P_1 \\ 0 & I & -P_2 \end{bmatrix}$$

```
[U,~] = eig(C_P,'complex');
X = U*kron(eye(n), ones(deg,1));
S = [X  C_P*X  C_P^(deg-1)*X];
C_R = S\C_P*S;
spy(abs(C_R)>1e-12)
```

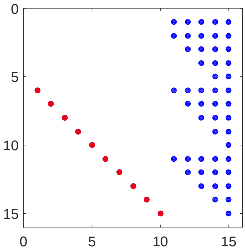


Hessenberg form

```
n = 5; deg = 3; % size and degree
P0 = randn(n); P1 = randn(n); P2 = randn(n); %
coefficient matrices
C_P = [ zeros(n) zeros(n) -P0; % left companion form
      eye(n) zeros(n) -P1;
     zeros(n) eye(n) -P2 ];
```

$$C_L(P) = \begin{bmatrix} 0 & 0 & -P_0 \\ I & 0 & -P_1 \\ 0 & I & -P_2 \end{bmatrix}$$

```
[U,~] = hess(C_P);
X = U*kron(eye(n), eye(deg,1));
S = [X  C_P*X  C_P^(deg-1)*X];
C_R = S\C_P*S;
spy(abs(C_R)>1e-12)
```



Standard Pairs

$$P(\lambda) = I_n\lambda^\ell + P_{\ell-1}\lambda^{\ell-1} + \ldots + P_1\lambda + P_0 \in \mathbb{F}[\lambda]^{n\times n}$$

G-L-R[4]: If $A \in \mathbb{F}^{n\ell\times n\ell}$ is a linearization of $P(\lambda)$, there is $X \in \mathbb{F}^{n\ell\times n}$ such that

- $S = \begin{bmatrix} X & AX & \cdots & A^{\ell-1}X \end{bmatrix}$ invertible,
- $S^{-1}AS = \begin{bmatrix} & & & -P_0 \\ I_n & & & -P_1 \\ & \ddots & & \vdots \\ & & I_n & -P_{\ell-1} \end{bmatrix} =: C_L(P)$

[4] Gohberg, Lancaster, Rodman: *Matrix Polynomials*

General idea

Given a monic $P(\lambda)$

- Pick up a monic linearization $\lambda I_{n\ell} - A$ of $P(\lambda)$
- Find $X \in \mathbb{F}^{n\ell \times n}$ such that
  - $S = \begin{bmatrix} X & AX & \cdots & A^{\ell-1}X \end{bmatrix}$ invertible,
  - $S^{-1}AS = \begin{bmatrix} & & & -T_0 \\ I_n & & & -T_1 \\ & \ddots & & \vdots \\ & & I_n & -T_{\ell-1} \end{bmatrix}$

    T_i upper triangular
- Define $T(\lambda) = I_n\lambda^\ell + T_{\ell-1}\lambda^{\ell-1} + \ldots + T_1\lambda + T_0$

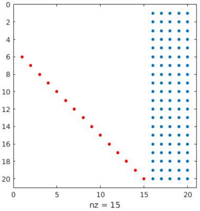
<u>Conclusion</u>:

$T(\lambda)$ is upper triangular with the same elementary divisors as $P(\lambda)$.

Substitute *triangular* by *block-diagonal* or *Hessenberg* for other simpler forms

Controller form

Π^T



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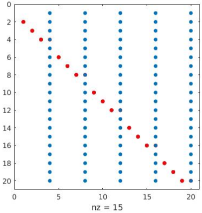
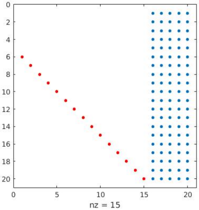


$$\left[\begin{array}{ccc|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & p^1_{11} & p^1_{12} & p^1_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & p^1_{21} & p^1_{22} & p^1_{23} \\
0 & 0 & 0 & 0 & 0 & 0 & p^1_{31} & p^1_{32} & p^1_{33} \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & p^2_{11} & p^2_{12} & p^2_{13} \\
0 & 1 & 0 & 0 & 0 & 0 & p^2_{21} & p^2_{22} & p^2_{23} \\
0 & 0 & 1 & 0 & 0 & 0 & p^2_{31} & p^2_{32} & p^2_{33} \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & p^3_{11} & p^3_{12} & p^3_{13} \\
0 & 0 & 0 & 0 & 1 & 0 & p^3_{21} & p^3_{22} & p^3_{23} \\
0 & 0 & 0 & 0 & 0 & 1 & p^3_{31} & p^3_{32} & p^3_{33}
\end{array}\right]
\longrightarrow
\left[\begin{array}{ccc|ccc|ccc}
0 & 0 & p^1_{11} & 0 & 0 & p^1_{12} & 0 & 0 & p^1_{13} \\
1 & 0 & p^2_{11} & 0 & 0 & p^2_{12} & 0 & 0 & p^2_{13} \\
0 & 1 & p^3_{11} & 0 & 0 & p^3_{12} & 0 & 0 & p^3_{13} \\
\hline
0 & 0 & p^1_{21} & 0 & 0 & p^1_{22} & 0 & 0 & p^1_{23} \\
0 & 0 & p^2_{21} & 1 & 0 & p^2_{22} & 0 & 0 & p^2_{23} \\
0 & 0 & p^3_{21} & 0 & 1 & p^3_{22} & 0 & 0 & p^3_{23} \\
\hline
0 & 0 & p^1_{31} & 0 & 0 & p^1_{32} & 0 & 0 & p^1_{33} \\
0 & 0 & p^2_{31} & 0 & 0 & p^2_{32} & 1 & 0 & p^2_{33} \\
0 & 0 & p^3_{31} & 0 & 0 & p^3_{32} & 0 & 1 & p^3_{33}
\end{array}\right]$$

$$\begin{bmatrix} x_1 & x_2 & x_3 & Ax_1 & Ax_2 & Ax_3 & A^2x_1 & A^2x_2 & A^2x_3 \end{bmatrix}$$
$$\updownarrow$$
$$\begin{bmatrix} x_1 & Ax_1 & A^2x_1 & x_2 & Ax_2 & A^2x_2 & x_3 & Ax_3 & A^2x_3 \end{bmatrix}$$

Controller form

Π^T



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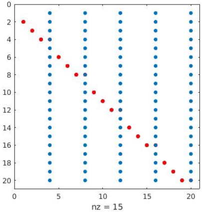


$$
\left[\begin{array}{ccc|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & p^1_{11} & p^1_{12} & p^1_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & {\color{red}0} & p^1_{22} & p^1_{23} \\
0 & 0 & 0 & 0 & 0 & 0 & {\color{red}0} & {\color{red}0} & p^1_{33} \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & p^2_{11} & p^2_{12} & p^2_{13} \\
0 & 1 & 0 & 0 & 0 & 0 & {\color{red}0} & p^2_{22} & p^2_{23} \\
0 & 0 & 1 & 0 & 0 & 0 & {\color{red}0} & {\color{red}0} & p^2_{33} \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & p^3_{11} & p^3_{12} & p^3_{13} \\
0 & 0 & 0 & 0 & 1 & 0 & {\color{red}0} & p^3_{22} & p^3_{23} \\
0 & 0 & 0 & 0 & 0 & 1 & {\color{red}0} & {\color{red}0} & p^3_{33}
\end{array}\right]
\longrightarrow
\left[\begin{array}{ccc|ccc|ccc}
0 & 0 & p^1_{11} & 0 & 0 & p^1_{12} & 0 & 0 & p^1_{13} \\
1 & 0 & p^2_{11} & 0 & 0 & p^2_{12} & 0 & 0 & p^2_{13} \\
0 & 1 & p^3_{11} & 0 & 0 & p^3_{12} & 0 & 0 & p^3_{13} \\
\hline
0 & 0 & p^1_{21} & 0 & 0 & p^1_{22} & 0 & 0 & p^1_{23} \\
0 & 0 & p^2_{21} & 1 & 0 & p^2_{22} & 0 & 0 & p^2_{23} \\
0 & 0 & p^3_{21} & 0 & 1 & p^3_{22} & 0 & 0 & p^3_{23} \\
\hline
0 & 0 & p^1_{31} & 0 & 0 & p^1_{32} & 0 & 0 & p^1_{33} \\
0 & 0 & p^2_{31} & 0 & 0 & p^2_{32} & 1 & 0 & p^2_{33} \\
0 & 0 & p^3_{31} & 0 & 0 & p^3_{32} & 0 & 1 & p^3_{33}
\end{array}\right]
$$

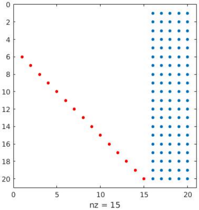
$$
\begin{bmatrix} x_1 & x_2 & x_3 & Ax_1 & Ax_2 & Ax_3 & A^2x_1 & A^2x_2 & A^2x_3 \end{bmatrix}
$$

$$\updownarrow$$

$$
\begin{bmatrix} x_1 & Ax_1 & A^2x_1 & x_2 & Ax_2 & A^2x_2 & x_3 & Ax_3 & A^2x_3 \end{bmatrix}
$$

Controller form

Π^T



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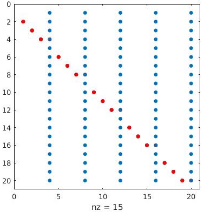


$$\left[\begin{array}{ccc|ccc|ccc} 0&0&0&0&0&0&p^1_{11}&p^1_{12}&p^1_{13}\\ 0&0&0&0&0&0&{\color{red}0}&p^1_{22}&p^1_{23}\\ 0&0&0&0&0&0&{\color{red}0}&{\color{red}0}&p^1_{33}\\ \hline 1&0&0&0&0&0&p^2_{11}&p^2_{12}&p^2_{13}\\ 0&1&0&0&0&0&{\color{red}0}&p^2_{22}&p^2_{23}\\ 0&0&1&0&0&0&{\color{red}0}&{\color{red}0}&p^2_{33}\\ \hline 0&0&0&1&0&0&p^3_{11}&p^3_{12}&p^3_{13}\\ 0&0&0&0&1&0&{\color{red}0}&p^3_{22}&p^3_{23}\\ 0&0&0&0&0&1&{\color{red}0}&{\color{red}0}&p^3_{33} \end{array}\right] \longleftrightarrow \left[\begin{array}{ccc|ccc|ccc} 0&0&p^1_{11}&0&0&p^1_{12}&0&0&p^1_{13}\\ 1&0&p^2_{11}&0&0&p^2_{12}&0&0&p^2_{13}\\ 0&1&p^3_{11}&0&0&p^3_{12}&0&0&p^3_{13}\\ \hline 0&0&{\color{red}0}&0&0&p^1_{22}&0&0&p^1_{23}\\ 0&0&{\color{red}0}&1&0&p^2_{22}&0&0&p^2_{23}\\ 0&0&{\color{red}0}&0&1&p^3_{22}&0&0&p^3_{23}\\ \hline 0&0&{\color{red}0}&0&0&{\color{red}0}&0&0&p^1_{33}\\ 0&0&{\color{red}0}&0&0&{\color{red}0}&1&0&p^2_{33}\\ 0&0&{\color{red}0}&0&0&{\color{red}0}&0&1&p^3_{33} \end{array}\right]$$

$$\begin{bmatrix} x_1 & x_2 & x_3 & Ax_1 & Ax_2 & Ax_3 & A^2x_1 & A^2x_2 & A^2x_3 \end{bmatrix}$$

$$\updownarrow$$

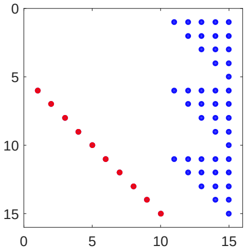
$$\begin{bmatrix} x_1 & Ax_1 & A^2x_1 & x_2 & Ax_2 & A^2x_2 & x_3 & Ax_3 & A^2x_3 \end{bmatrix}$$

$$< x_1, Ax_1, A^2x_1 >, \quad < x_1, Ax_1, A^2x_1, x_2, Ax_2, A^2x_2 >$$

A-invariant subspaces

Triangular form

```
n = 5; deg = 3; % size and degree
P0 = randn(n); P1 = randn(n); P2 = randn(n); %
coefficient matrices
C_P = [ zeros(n) zeros(n) -P0; % left companion form
      eye(n) zeros(n) -P1;
      zeros(n) eye(n) -P2 ];
[U,~] = schur(C_P,'complex');
X = U*kron(eye(n), ones(deg,1));
S = [X  C_P*X  C_P^(deg-1)*X];
C_R = S\C_P*S;
spy(abs(C_R)>1e-12)
```



Reduction to triangular form: First step

$P(\lambda) = I_n\lambda^\ell + P_{\ell-1}\lambda^{\ell-1} + \ldots + P_1\lambda + P_0 \in \mathbb{F}[\lambda]^{n\times n}$

$A \in \mathbb{F}^{n\ell\times n\ell}$= linearization of $P(\lambda)$

- Compute a Schur form (real or complex according as $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$)

$$U^*AU = \begin{bmatrix}
x & x & \cdots & x & x & x & \cdots & x & & x & x & \cdots & x \\
 & x & \cdots & x & x & x & \cdots & x & & x & x & \cdots & x \\
 & & \ddots & \vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & & \vdots \\
 & & & x & x & x & \cdots & x & & x & x & \cdots & x \\
 & & & & x & x & \cdots & x & & x & x & \cdots & x \\
 & & & & & x & \cdots & x & & x & x & \cdots & x \\
 & & & & & & \ddots & \vdots & \cdots & \vdots & \vdots & & \vdots \\
 & & & & & & & x & & x & x & \cdots & x \\
 & & & & & & & & \ddots & & & & \\
 & & & & & & & & & x & x & \cdots & x \\
 & & & & & & & & & & x & \cdots & x \\
 & & & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & & & & x
\end{bmatrix}$$

$$n\ell \times n\ell$$

Reduction to triangular form: First step

$P(\lambda) = I_n\lambda^\ell + P_{\ell-1}\lambda^{\ell-1} + \ldots + P_1\lambda + P_0 \in \mathbb{F}[\lambda]^{n\times n}$

$A \in \mathbb{F}^{n\ell\times n\ell}$= linearization of $P(\lambda)$

- Compute a Schur form (real or complex according as $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$)

$$U^*AU = \underbrace{\left[\begin{array}{cccc|cccc|c|cccc}
x & x & \cdots & x & x & x & \cdots & x & & x & x & \cdots & x \\
 & x & \cdots & x & x & x & \cdots & x & & x & x & \cdots & x \\
 & & \ddots & \vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & & \vdots \\
 & & & x & x & x & \cdots & x & & x & x & \cdots & x \\
\hline
 & & & & x & x & \cdots & x & & x & x & \cdots & x \\
 & & & & & x & \cdots & x & & x & x & \cdots & x \\
 & & & & & & \ddots & \vdots & \cdots & \vdots & \vdots & & \vdots \\
 & & & & & & & x & & x & x & \cdots & x \\
\hline
 & & & & & & & & \ddots & & & & \\
\hline
 & & & & & & & & & x & x & \cdots & x \\
 & & & & & & & & & & x & \cdots & x \\
 & & & & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & & & & x
\end{array}\right]}_{n\ell\times n\ell}$$

Nonderogatory $\ell \times \ell$ blocks?

Schur form for linearizations of matrix polynomials

If $\lambda I - A$ is a linearization of a monic $P(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$, the geometric multiplicity of the eigenvalues of A cannot be greater than n.

Theorem

Let $A \in \mathbb{F}^{\ell n\times \ell n}$ be a matrix whose eigenvalues have geometric multiplicity at most n. Then A has a Schur decomposition

$$A = Q \begin{bmatrix} T_{11} & * & * & * \\ & T_{22} & * & * \\ & & \ddots & * \\ & & & T_{rr} \end{bmatrix} Q^H,$$

where:

(i) *if $\mathbb{F} = \mathbb{C}$ then $r = n$ and $T_{ii} \in \mathbb{C}^{\ell\times\ell}$ are non-derogatory,*

(ii) *if $\mathbb{F} = \mathbb{R}$ then each T_{ii} is real and either of size $\ell \times \ell$ and non-derogatory or of size $2\ell \times 2\ell$ and such that all eigenvalues have geometric multiplicity 1 or 2.*

A numerically stable procedure to construct the Schur form (Complex case)

Algebraic multiplicity $\leq n$

Example: $n = 3$, $\ell = 2$

Bai-Demmel

$$A = \begin{bmatrix} 1 & 0 & 1 & \frac{\sqrt{2}(-1+\mathrm{i})}{2} & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{\sqrt{3}(-1-\mathrm{i})}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{\sqrt{2}(1-\mathrm{i})}{2} & 0 \\ 0 & 0 & 0 & -\mathrm{i} & 0 & \frac{\sqrt{2}(-1-\mathrm{i})}{2} \\ 0 & 0 & 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{i} \end{bmatrix} \xrightarrow{\texttt{[U1,T1]= ordschur (eye}(n)\texttt{)}, A, [1\,0\,0\,0\,1\,0]\texttt{)}}$$

A numerically stable procedure to construct the Schur form (Complex case)

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Bai-Demmel

$$A = \begin{bmatrix} 1 & 0 & 1 & \frac{\sqrt{2}(-1+\mathrm{i})}{2} & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{\sqrt{3}(-1-\mathrm{i})}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{\sqrt{2}(1-\mathrm{i})}{2} & 0 \\ 0 & 0 & 0 & -\mathrm{i} & 0 & \frac{\sqrt{2}(-1-\mathrm{i})}{2} \\ 0 & 0 & 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{i} \end{bmatrix} \xrightarrow{\texttt{[U1,T1]= ordschur(eye}(n)\texttt{,}A\texttt{,}[1\,0\,0\,0\,1\,0]\texttt{)}}$$

$$T_1 = \begin{bmatrix} 1 & \frac{\sqrt{2}}{3}\mathrm{i} & \frac{1}{3} & \frac{\sqrt{6}}{3}\mathrm{i} & -\frac{\sqrt{2}(1+\mathrm{i})}{2} & 1 \\ 0 & \mathrm{i} & \frac{\sqrt{2}(1+\mathrm{i})}{2} & \frac{\sqrt{3}(1-\mathrm{i})}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{i} & \frac{\sqrt{2}(-1+\mathrm{i})}{2} \\ 0 & 0 & 0 & 0 & 0 & \mathrm{i} \end{bmatrix} \xrightarrow{\texttt{[U2,T2]= ordschur(U1,T1,}[1\,1\,1\,0\,1\,0]}$$

A numerically stable procedure to construct the Schur form (Complex case)

Algebraic multiplicity $\leq n$

Bai-Demmel

Example: $n = 3$, $\ell = 2$

$$A = \begin{bmatrix} 1 & 0 & 1 & \frac{\sqrt{2}(-1+i)}{2} & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{\sqrt{3}(-1-i)}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{\sqrt{2}(1-i)}{2} & 0 \\ 0 & 0 & 0 & -i & 0 & \frac{\sqrt{2}(-1-i)}{2} \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & i \end{bmatrix} \xrightarrow{\texttt{[U1,T1]= ordschur (eye}(n)\texttt{)}, A, [1\,0\,0\,0\,1\,0]\texttt{)}}$$

$$T_1 = \begin{bmatrix} 1 & \frac{\sqrt{2}}{3}i & \frac{1}{3} & \frac{\sqrt{6}}{3}i & -\frac{\sqrt{2}(1+i)}{2} & 1 \\ 0 & i & \frac{\sqrt{2}(1+i)}{2} & \frac{\sqrt{3}(1-i)}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & \frac{\sqrt{2}(-1+i)}{2} \\ 0 & 0 & 0 & 0 & 0 & i \end{bmatrix} \xrightarrow{\texttt{[U2,T2]= ordschur (U1,T1,}[1\,1\,1\,0\,1\,0]}$$

$$T_2 = \begin{bmatrix} 1 & \frac{\sqrt{2}}{3}i & \frac{1}{3} & i & \frac{\sqrt{3}(1+i)}{3} & 1 \\ 0 & i & \frac{\sqrt{2}(1+i)}{2} & 0 & -\frac{\sqrt{6}}{2}i & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & -i \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \end{bmatrix}$$

Some observations

- The real case is much more involved: combine 2×2-blocks. Also the diagonal blocks of the Schur form of A:

$$A = Q \begin{bmatrix} T_{11} & * & * & * \\ & T_{22} & * & * \\ & & \ddots & * \\ & & & T_{rr} \end{bmatrix} Q^H,$$

  may have size $2\ell \times 2\ell$.
  Example: $n = 2$, $\ell = 3$

$$A = \mathrm{Diag}\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \right).$$

- The case when there are eigenvalues with algebraic multiplicity $> n$ is open except for quadratic matrix polynomials. It requires a different approach.

From the Schur form to the triangular controller form

$$A = QTQ^H \text{ with } T = \begin{bmatrix} T_{11} & * & * & * \\ & T_{22} & * & * \\ & & \ddots & * \\ & & & T_{rr} \end{bmatrix}, T_{ii} \in \mathbb{F}^{\ell\times\ell} \text{ non-derogatory}$$

or $T_{ii} \in \mathbb{R}^{2\ell\times 2\ell}$ with eigenvalues of geometric multiplicity 1 or 2.

Lemma

If all eigenvalues of $A \in \mathbb{F}^{k\ell\times k\ell}$ have geometric multiplicity at most k then $\exists X \in \mathbb{F}^{k\ell\times k}$ such that $\begin{bmatrix} X & AX & \cdots & A^{\ell-1}X \end{bmatrix}$ is nonsingular.

Apply Lemma to T_{ii}: $\exists V_i$ ($V_i = [x_i]$ if $T_{ii} \in \mathbb{F}^{\ell\times\ell}$ and $V_i = [x_i\ y_i]$ if $T_{ii} \in \mathbb{R}^{2\ell\times 2\ell}$) such that $\begin{bmatrix} V_i & T_{ii}V_i & \cdots & T_{ii}^{\ell-1}V_i \end{bmatrix}$ nonsingular.

From the Schur form to the triangular controller form

- $T_{ii} \in \mathbb{F}^{\ell \times \ell}$ non-derogatory:

$$Z_i = \begin{bmatrix} x_i & T_{ii}x_i & \cdots & T_{ii}^{\ell}x_i \end{bmatrix}$$

$$Z_i^{-1} T_{ii} Z_i = \begin{bmatrix} & & & & * \\ 1 & & & & * \\ & 1 & & & * \\ & & \ddots & & \\ & & & 1 & * \end{bmatrix}$$

- $T_{ii} \in \mathbb{R}^{2\ell \times 2\ell}$ with e. v. of geometric multiplicity 1 or 2:

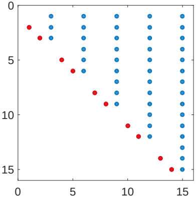
$$Z_i = \begin{bmatrix} x_i & T_{ii}x_i & \cdots & T_{ii}^{\ell}x_i & y_i & T_{ii}y_i & \cdots T_{ii}^{\ell}y_i \end{bmatrix}$$

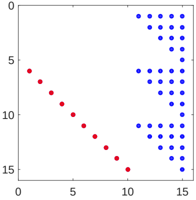
$$Z_i^{-1} T_{ii} Z_i = \left[\begin{array}{cccc|cccc} & & & * & & & & * \\ 1 & & & * & & & & * \\ & \ddots & & & & & & \\ & & 1 & * & & & & * \\ \hline & & & * & & & & * \\ & & & * & 1 & & & * \\ & & & & & \ddots & & \\ & & & * & & & & \\ & & & * & & & 1 & * \end{array}\right]$$

From the Schur form to the triangular companion form

$$Z = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_r$$

then Z is nonsingular and $Z^{-1}TZ = Z^{-1}Q^H AQZ$ is upper (block-)triangular in controller form.

Π^T 

Π = 

Π^T Z $\Pi = \begin{bmatrix} W & TW & \cdots & T^{\ell-1}W \end{bmatrix}$

$W = V_1 \oplus V_2 \oplus \cdots \oplus V_r$, $V_i = [x_i]$ or $V_i = [x_i \; y_i]$.

$$Q \begin{bmatrix} W & TW & \cdots & T^{\ell-1}W \end{bmatrix} = \begin{bmatrix} X & AX & \cdots & A^{\ell-1}X \end{bmatrix}$$

Summarizing

Given $P(\lambda) \in \mathbb{F}^{n\times n}$ of degree ℓ, let $A \in \mathbb{F}^{n\ell\times n\ell}$ be a linearization.

1. Compute a Schur form S of A
2. Reorder S to produce $Q^H A Q = T = \begin{bmatrix} T_{11} & * & * & * \\ & T_{22} & * & * \\ & & \ddots & * \\ & & & T_{rr} \end{bmatrix}$,

   $T_{ii} \in \mathbb{F}^{\ell\times\ell}$ non-derogatory or $T_{ii} \in \mathbb{R}^{2\ell\times 2\ell}$ with eigenvalues of geometric multiplicity 1 or 2.
3. Obtain $V_i = [x_i]$ or $V_i = [x_i \;\; y_i]$ so that $\begin{bmatrix} V_i & T_{ii}V_i & \cdots & T_{ii}^{\ell-1}V_i \end{bmatrix}$ is nonsingular
4. Put $W = \mathrm{Diag}(V_1, \dots, V_r)$, define $X = QW$ and compute $B = \begin{bmatrix} X & AX & \cdots & A^{\ell-1}X \end{bmatrix}^{-1} A \begin{bmatrix} X & AX & \cdots & A^{\ell-1}X \end{bmatrix}$

B is the left companion form of an upper (block-)triangular matrix polynomial with the same elementary divisors as $P(\lambda)$

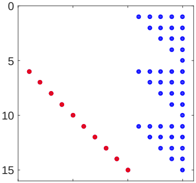
<u>Remark</u>:

- In practice X can be taken with orthonormal columns.

Trinagular form: Generic case

```
n = 5; deg = 3;
P0 = randn(n); P1 = randn(n); P2 = randn(n);
C_P = [ zeros(n) zeros(n) -P0;
      eye(n) zeros(n) -P1;
      zeros(n) eye(n) -P2 ];
[U,~] = schur(C_P,'complex');
X = U*kron(eye(n), ones(deg,1));
S = [X  C_P*X C_P^(deg-1)*X];
C_R = S\C_P*S;
spy(abs(C_R)>1e-12)
```

`kron(eye(n), ones(deg,1))` $=\mathrm{Diag}\left(\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix},\ldots,\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}\right).$



Hessenberg form

Given a monic $P(\lambda) \in \mathbb{F}^{n\times n}$ of degree ℓ, $A \in \mathbb{F}^{n\ell \times n\ell}$ linearization.

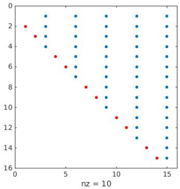
1. Compute a Hessenberg form of A:

$$Q^H A Q = H = \begin{bmatrix} H_{11} & * & * & * \\ & H_{22} & * & * \\ & & \ddots & * \\ & & & H_{nn} \end{bmatrix},$$ and assume that

   $H_{ii} \in \mathbb{F}^{\ell\times\ell}$ is unreduced (the $(k+1,k)$ entry $\neq 0$).

2. Put $v_i = e_{(i-1)\ell+1}$. Then $Z_i = K_\ell(H_i, v_i)$ is nonsingular, and if $Z = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_n$

   $Z^{-1}HZ$ =

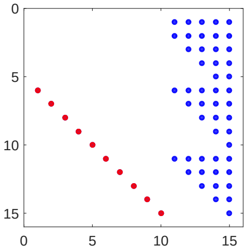


3. Put $W = \mathrm{Diag}(v_1, \ldots, v_n)$, define $X = QW$ and compute $B = K_\ell(A,X)^{-1} A K_\ell(A,X)$

B is the left companion form of an upper Hessenberg matrix polynomial with the same elementary divisors as $P(\lambda)$

Hessenberg form: Generic case

```
n = 5; deg = 3; % size and degree
P0 = randn(n); P1 = randn(n); P2 = randn(n); %
coefficient matrices
C_P = [ zeros(n) zeros(n) -P0; % left companion form
     eye(n) zeros(n) -P1;
     zeros(n) eye(n) -P2 ];
[U,~] = hess(C_P);
X = U*kron(eye(n), eye(deg,1));
S = [X  C_P*X  C_P^(deg-1)*X];
C_R = S\C_P*S;
spy(abs(C_R)>1e-12)
```



Concluding Remarks: Open Problems

- Schur form with non-derogatory $\ell \times \ell$ diagonal blocks (or, if $\mathbb{F} = \mathbb{R}$, $2\ell \times 2\ell$ with all eigenvalues of geometric multiplicity 1 or 2) when A has eigenvalues of algebraic multiplicity $> n$.
- How to manage the case when the $\ell \times \ell$ diagonal blocks of the Hessenberg form of A are not unreduced? Is it possible to move zeros on the sub-diagonal by using Givens rotations or Householder reflectors?
- How to compute the generating vectors?
- Stability analysis of the companion preserving transformations.

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Thank you