The sepr-sequence of a Hermitian matrix

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Outline

Basic terminology

Let B be $n \times n$ matrix, and let $\alpha, \beta \subseteq [n]$.

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2. $\det B[\alpha, \beta]$ is a *principal* minor if $\alpha = \beta$.
3. The *order* of the principal minor $\det B[\alpha]$ is $|\alpha|$.

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BUT, $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ is **NOT** a principal submatrix.

Applications of principal minors

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- matrix theory;
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In particular, as stated by [Griffin and Tsatsomeros, 2006]:

- detection of P-matrices;
- cartan matrices in Lie algebras;
- univalent differentiable mappings;
- self-validating algorithms;
- interval matrix analysis;
- counting of spanning trees of a graph using the Laplacian;
- D-nilpotent automorphisms;
- inverse multiplicative eigenvalue problem.

The principal minor assignment problem [Holtz and Schneider, 2002]

Given a vector $\mathbf{u} \in \mathbb{R}^{2^n-1}$, when is there an $n \times n$ matrix having its $2^n - 1$ principal minors given by the entries of $\mathbf{u}$?

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Our focus here will be on **Hermitian** matrices.

Example

Let $\mathbf{u} = [\underbrace{\mathbf{1},\ \mathbf{4},\ \mathbf{6}}_{Order1},\ \underbrace{\mathbf{0},\ -\mathbf{3},\ -\mathbf{1}}_{Order2},\ \underbrace{-\mathbf{1}}_{Order3}]^T \in \mathbb{R}^{2^3-1}$.

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$$B = \begin{bmatrix} \mathbf{1} & 2 & 3 \\ 2 & \mathbf{4} & 5 \\ 3 & 5 & \mathbf{6} \end{bmatrix}.$$

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$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}.$$

$$\det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0; \ \det \begin{bmatrix} 1 & 3 \\ 3 & 6 \end{bmatrix} = -3; \ \det \begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} = -1.$$

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$\det B = -1$.

Example

Consider

$$\mathbf{u} = [\underbrace{\mathbf{0}, \ \mathbf{a}, \ \mathbf{b}}_{Order1}, \underbrace{\mathbf{0}, \ \mathbf{0}, \ \mathbf{0}}_{Order2}, \underbrace{\ \mathbf{c}\ }_{Order3}]^T \in \mathbb{R}^{2^3-1},$$

where $a, b, c \neq 0$.

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where $a, b, c \neq 0$.

There is **<u>not</u>** any 3×3 Hermitian matrix having its eight principal minors given by $\mathbf{u}$.

The epr-sequence

Definition (Butler et al.; 2016)

The *enhanced principal rank characteristic sequence* of an $n \times n$ matrix B is the sequence (*epr-sequence*) $\text{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$, where

$$\ell_k = \begin{cases} \texttt{A} & \text{if } \textbf{all} \text{ order-}k \text{ principal minors are nonzero;} \\ \texttt{N} & \text{if } \textbf{none} \text{ of the order-}k \text{ principal minors are nonzero;} \\ \texttt{S} & \text{if } \textbf{some} \text{ (}\underline{\text{but not all}}\text{) order-}k \text{ principal minors are nonzero.} \end{cases}$$

Example

Let $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$ and let $\text{epr}(B) = \ell_1\ell_2\ell_3\ell_4$.

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$\det(B[\{1,2,3\}]) = \mathbf{0}$ and $\det(B[\{2,3,4\}]) \neq \mathbf{0} \implies \ell_3 = \mathtt{S}$.

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None of the order-2 principal minors are nonzero $\implies \ell_{\mathbf{2}} = \mathtt{N}$.

$\det(B[\{1,2,3\}]) = \mathbf{0}$ and $\det(B[\{2,3,4\}]) \neq \mathbf{0} \implies \ell_{\mathbf{3}} = \mathtt{S}$.

$\det(\mathbf{B}) = \mathbf{0} \implies \ell_{\mathbf{4}} = \mathtt{N}$.

Hence, $\operatorname{epr}(B) = \mathtt{ANSN}$.

The Inverse Theorem

Theorem (Butler et al.; 2016)

Suppose B is an $n \times n$ nonsingular Hermitian matrix.
If $\operatorname{epr}(B) = \ell_1\ell_2 \cdots \ell_{n-1}\mathtt{A}$, *then* $\operatorname{epr}(B^{-1}) = \ell_{n-1}\ell_{n-2} \cdots \ell_1\mathtt{A}$.

NN forces Ns

Theorem (Butler et al.; 2016)

Suppose B is an $n \times n$ Hermitian matrix, $\operatorname{epr}(B) = \ell_1 \ell_2 \cdots \ell_n$, *and $\ell_k = \ell_{k+1} = \mathtt{N}$ for some k. Then $\ell_i = \mathtt{N}$ for all $i \geq k$.*

The sepr-sequence

Definition (Martínez; under review)

Let B be a Hermitian matrix with $\text{epr}(B) = \ell_1\ell_2\cdots\ell_n$. The *signed enhanced principal rank characteristic sequence* (*sepr-sequence*) of B is the sequence $\text{sepr}(B) = t_1t_2\cdots t_n$, where

$$t_k = \left\{ \right.$$

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Terminology & notation

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2. A sequence $x_1 x_2 \cdots x_k$ is *nonpositive* if $x_j \in \{\mathtt{A}^-, \mathtt{S}^-, \mathtt{N}\}$.
3. $\cdots \mathtt{X} \cdots \mathtt{Y} \cdots$.

Basic result

Proposition (Martínez; under review)

No Hermitian matrix can have any of the following sepr-sequences.

1. $\mathtt{A}^*\mathtt{A}^+\cdots$.
2. $\mathtt{A}^*\mathtt{S}^+\cdots$.
3. $\mathtt{A}^*\mathtt{N}\cdots$.
4. $\mathtt{S}^*\mathtt{A}^+\cdots$.
5. $\mathtt{S}^*\mathtt{S}^+\cdots$.
6. $\mathtt{S}^*\mathtt{N}\cdots$.
7. $\mathtt{S}^+\mathtt{A}^+\cdots$.
8. $\mathtt{S}^-\mathtt{A}^+\cdots$.
9. $\mathtt{N}\mathtt{A}^*\cdots$.
10. $\mathtt{N}\mathtt{A}^+\cdots$.
11. $\mathtt{N}\mathtt{S}^*\cdots$.
12. $\mathtt{N}\mathtt{S}^+\cdots$.

$\mathtt{A^*N}$ and $\mathtt{NA^*}$

Theorem (Martínez; under review)

The following sequences cannot occur in the sepr-sequence of a Hermitian matrix:

1. $\mathtt{A^*N}$;
2. $\mathtt{NA^*}$.

$\mathtt{A}^{+}\mathtt{X}\mathtt{A}^{+}$ & $\mathtt{A}^{-}\mathtt{X}\mathtt{A}^{-}$

Theorem (Martínez; under review)

If any of the sequences $\mathtt{A}^{+}\mathtt{X}\mathtt{A}^{+}$ *or* $\mathtt{A}^{-}\mathtt{X}\mathtt{A}^{-}$ *occurs in the sepr-sequence of a Hermitian matrix, then* $\mathtt{X} \in \{\mathtt{A}^{+}, \mathtt{A}^{-}\}$.

$\mathtt{S}^+\mathtt{XA}^+$ & $\mathtt{S}^-\mathtt{XA}^-$

Theorem (Martínez; under review)

If any of the sequences $\mathtt{S}^+\mathtt{XA}^+$ *or* $\mathtt{S}^-\mathtt{XA}^-$ *occurs in the sepr-sequence of a Hermitian matrix, then* $\mathtt{X} \in \{\mathtt{A}^+, \mathtt{A}^-\}$.

$\mathtt{A}^{+}\mathtt{XS}^{+}$ & $\mathtt{A}^{-}\mathtt{XS}^{-}$

$\mathtt{A^+XS^+}$ & $\mathtt{A^-XS^-}$

Theorem (Martínez; under review)

For any $\mathtt{X}$ *and for* $\mathtt{Y} \in \{\mathtt{A}^*, \mathtt{A}^+, \mathtt{A}^-\}$,
if any of the sepr-sequences
$\cdots \mathtt{A}^+\mathtt{XS}^+ \cdots \mathtt{Y} \cdots$ *or* $\cdots \mathtt{A}^-\mathtt{XS}^- \cdots \mathtt{Y} \cdots$
is attainable by a Hermitian matrix, then $\mathtt{X} \in \{\mathtt{A}^+, \mathtt{A}^-\}$.

Nonnegative sequences

Definition

A sequence $x_1 x_2 \cdots x_k$ is *nonnegative* if $x_i \in \{\mathtt{A}^+, \mathtt{S}^+, \mathtt{N}\}$.

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Theorem (Martínez; under review)

Let B be an $n \times n$ Hermitian matrix, and let $\sigma = x_1 x_2 \cdots x_k$ be a nonnegative subsequence of $\operatorname{sepr}(B)$*, where $2 \le k \le n$. Then $x_2 x_3 \cdots x_k = \overline{\mathtt{A}^+}\ \overline{\mathtt{S}^+}\ \overline{\mathtt{N}}$.*

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Corollary (Martínez; under review)

Let B be a (Hermitian) positive semidefinite matrix. Then $\operatorname{sepr}(B) = \overline{\mathtt{A}^+}\ \overline{\mathtt{S}^+}\ \overline{\mathtt{N}}$, *where $\overline{\mathtt{N}}$ is nonempty if $\overline{\mathtt{S}^+}$ is nonempty.*

Thanks!

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