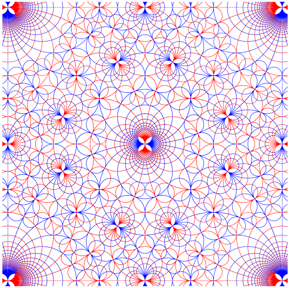
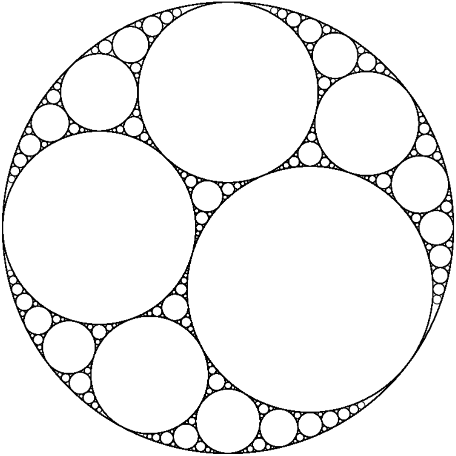
Circle packings, thin orbits and the arithmetic of imaginary quadratic fields

Katherine E. Stange

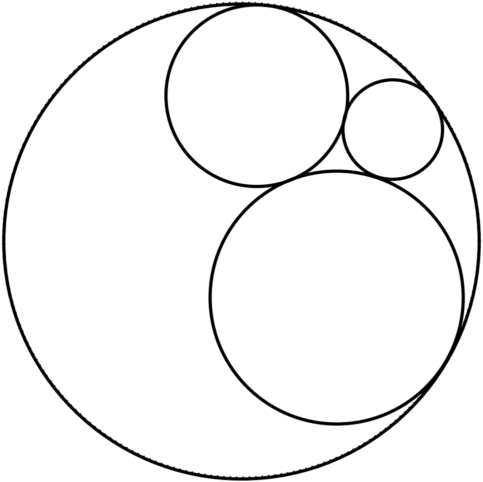


Alberta Number Theory Days, March 18, 2017

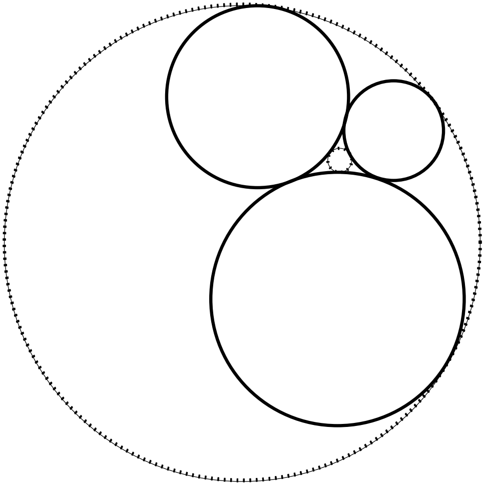
Apollonian Circle Packings



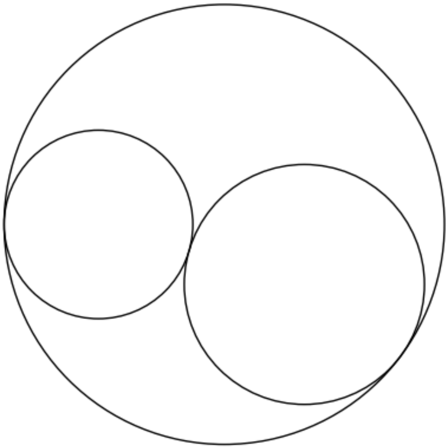
A Descartes quadruple is any collection of four circles which are pairwise mutually tangent, with disjoint interiors.



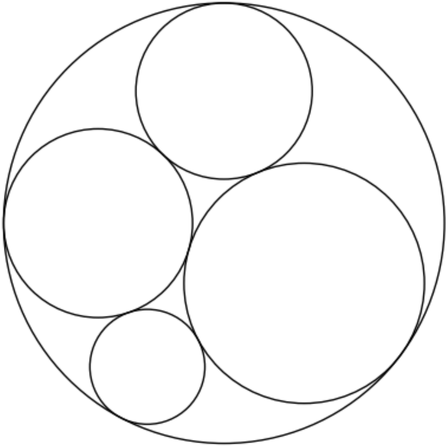
Given any three mutually tangent circles, there are exactly two ways to complete the triple to a Descartes quadruple.



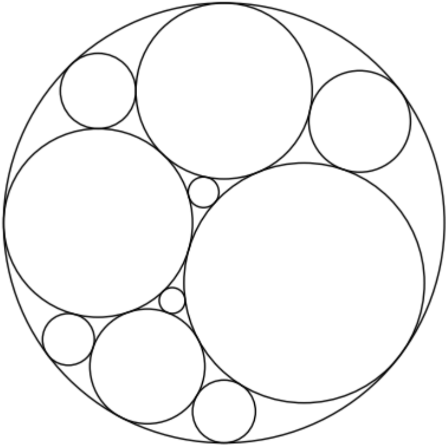
Beginning with any three mutually tangent circles...



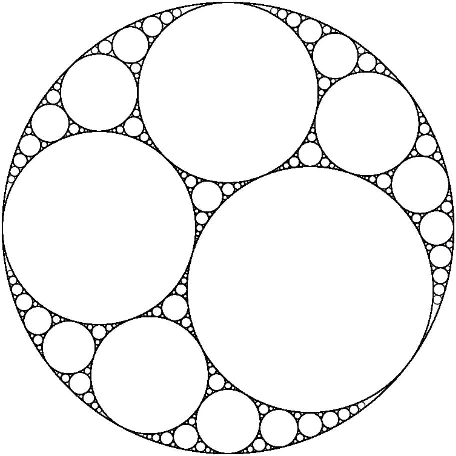
Beginning with any three mutually tangent circles, add in both new circles which would complete the triple to a Descartes quadruple.



Repeat: for every triple of mutually tangent circles in the collection, add the two 'completions.'



Repeating ad infinitum, we obtain an Apollonian circle packing.



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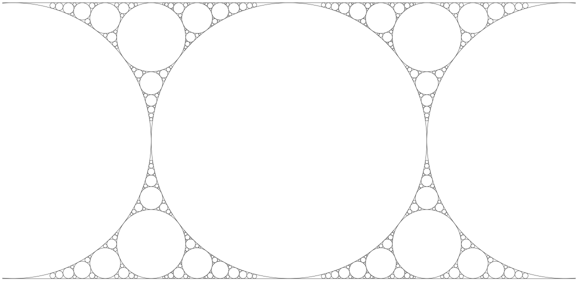
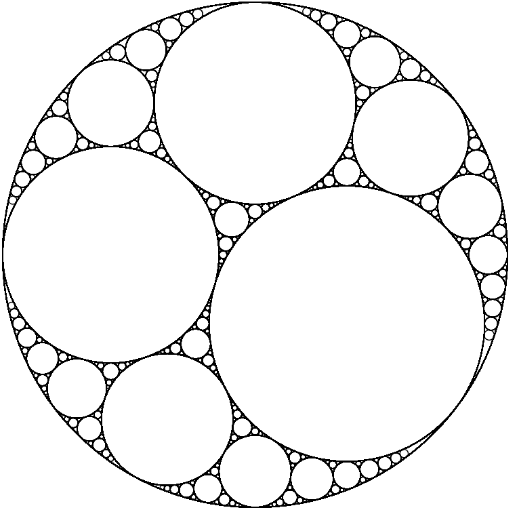
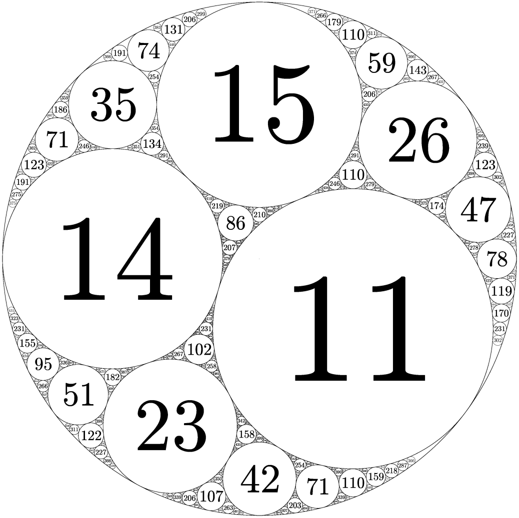






The outer circle has curvature -6 (its interior is outside).

The Descartes Rule

The curvatures (inverse radii) in a Descartes configuration satisfy

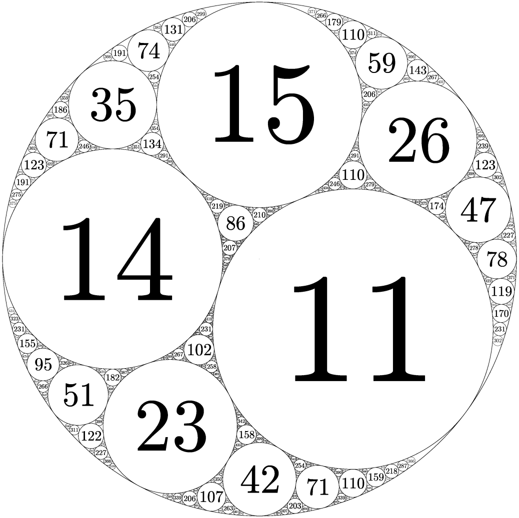
$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

If a, b, c are fixed, there are two solutions d, d', where

$$d + d' = 2(a + b + c).$$

Hence an **integer** Descartes quadruple generates an Apollonian packing of **integer curvatures.**

Which curvatures appear?



The outer circle has curvature -6 (its interior is outside).

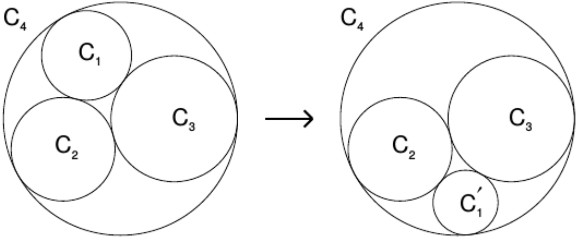
Local-Global Conjecture

Conjecture (Graham–Lagarias–Mallows–Wilks–Yan, Fuchs–Sanden)

$\mathcal{P}$ a primitive, integral ACP. Let S be the set of residues of curvatures modulo 24*. Then any sufficiently large integer with a residue in S occurs as a curvature.*

- Bourgain, Fuchs: Curvatures have positive density in $\mathbb{Z}$ (Sieve methods).
- Bourgain, Kontorovich: Density one occur (Circle method).

Apollonian group



(4 swaps)

$$\mathcal{A} = \left\langle S_1, S_2, S_3, S_4 : S_i^2 = 1 \right\rangle$$

Image from Graham, Lagarias, Mallows, Wilks, Yan

Apollonian group as Möbius transformations

$$\mathcal{A} \to PSL_2(\mathbb{Z}[i])$$

Möbius transformations:

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \leftrightarrow \left(z \mapsto \frac{\alpha z + \gamma}{\beta z + \delta} \right).$$

Möbius transformations act on the collection of circles in the extended complex plane.

The action extends to the upper half space, a model of hyperbolic space $\mathbb{H}^3$.

The exceptional isomorphism

$$\rho : \mathrm{PGL}_2(\mathbb{C}) \to \mathrm{SO}^+_{1,3}(\mathbb{R}).$$

- $\mathrm{SO}^+_{1,3}(\mathbb{R})$ acts on the 4D real vector space of Hermitian matrices,
$$\begin{pmatrix} b' & x+iy \\ x-iy & b \end{pmatrix}$$
preserving the determinant, a form of signature $1, 3$.
- $\mathrm{PGL}_2(\mathbb{C})$ acts by conjugation $\gamma \cdot M = \gamma^\dagger M \gamma$.
- Hermitian forms of determinant -1 (say) 'are' circles (take the zero set in $\widehat{\mathbb{C}}$). These form a hyperboloid in Minkowski space, a model of $\mathbb{H}^3$.
- The matrix above is a circle (b, b', x, y) of curvature b and curvature $\times$ center $x + iy$.

The Apollonian Group ($\mathbb{Z}[i]$)

Idea: act on *Descartes quadruples* instead of circles, coded as a 4×4 matrix

$$W_D = \begin{pmatrix} | & | & | & | \\ C_1 & C_2 & C_3 & C_4 \\ | & | & | & | \end{pmatrix}$$

Theorem (GLMWY)

C_1, C_2, C_3, C_4 form a Descartes configuration if and only if

$$W_D^{\dagger} G_M W_D = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

Codify swaps of Descartes quadruples as a matrix action:

$$W_D \mapsto W_D S_i, \quad i = 1, 2, 3, 4$$

The *Apollonian group* is $\langle S_1, S_2, S_3, S_4 \rangle \subset O_{3,1}(\mathbb{R})$.

Apollonian circle packing curvatures as a thin orbit

Thin group: $\mathcal{A}$ a subgroup of integer points of an algebraic group G, infinite index in $G(\mathbb{Z})$ yet Zariski dense in G.

Thin orbit: $\mathcal{A} \cdot \mathbf{v}_0$; very thin (slow growth of $|\mathbf{v}| < X$) compared to $G(\mathbb{Z}) \cdot \mathbf{v}_0$. For Apollonian group,

$$\#\{||A \cdot \mathbf{v}_0||_\infty < X\} \sim X^{1.30568...}$$

(Boyd, Kontorovich-Oh) Compare to X^2 for an $O_{3,1}(\mathbb{Z})$ orbit.

Curvatures: $\mathcal{C} := \{\langle \mathbf{w}, \mathcal{A} \cdot \mathbf{v}_0 \rangle\}$.

Phenomenon: $\mathcal{C}$ includes all but finitely many integers satisfying some congruence conditions.

Other examples (Kontorovich, Bulletin AMS): Zaremba's conjecture, Pythagorean triples

Local Obstructions and Strong Approximation

$$\mathcal{A} \to SL_2(\mathbb{Z}[i])$$

Reduction modulo m:

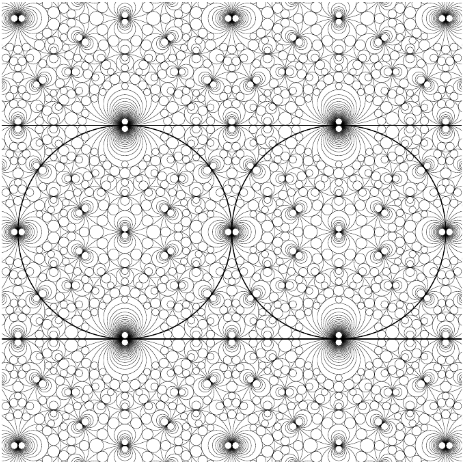
$$\rho_m : \mathcal{A} \to \mathcal{A}/\mathcal{A}(m)$$

Good primes, and good powers of bad primes:

$$\rho_{p^k}\mathcal{A} = \rho_{p^k} SL_2(\mathbb{Z}[i])$$

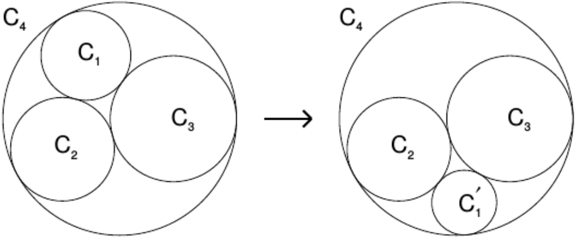
Bad prime powers for Apollonian group: 2,4,8,16,3.

The 'Superpacking'

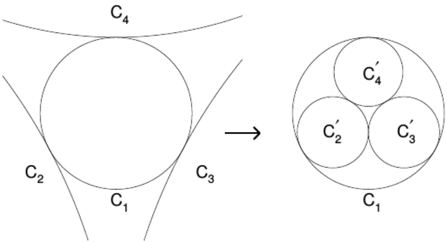


- Graham, Lagarias, Mallows, Wilks, Yan, *Apollonian Circle Packings: Geometry and Group Theory II. Super-Apollonian Group and Integral Packings*, Discrete and Computational Geometry, 2006.

Super-Apollonian group



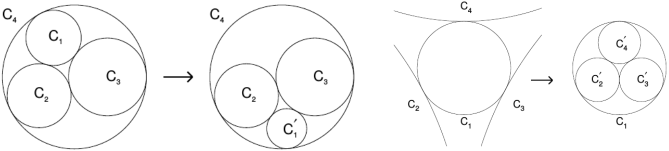
(4 swaps)



(4 inversions)

Images from Graham, Lagarias, Mallows, Wilks, Yan

Super-Apollonian group



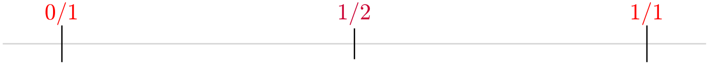
$$\Big\langle S_1, S_2, S_3, S_4, S_1^{\perp}, S_2^{\perp}, S_3^{\perp}, S_4^{\perp} :$$
$$S_i^2 = (S_i^{\perp})^2 = (S_i S_j^{\perp})^2 = (S_j^{\perp} S_i)^2 = 1 \Big\rangle$$

Index 48 in $PSL_2(\mathbb{Z}[i])$.

The Farey subdivision

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

The Farey subdivision



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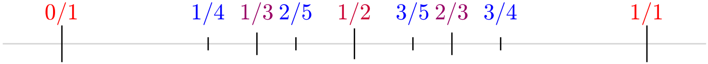


$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

Farey subdivision: frothy version

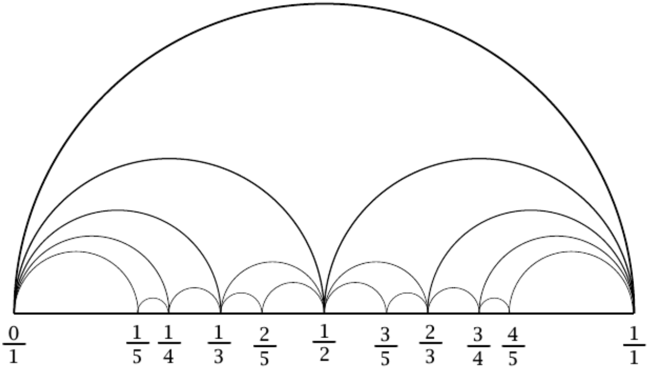
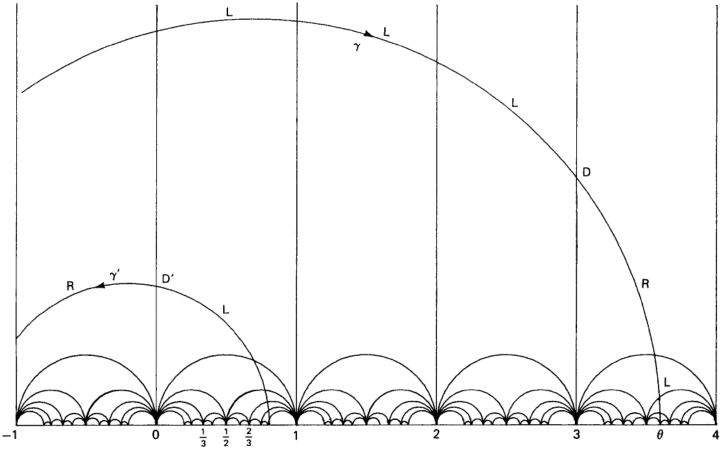


image from Allen Hatcher's *Topology of Numbers*

Continued fractions as geodesics



address of $\alpha = L^{a_0} R^{a_1} L^{a_2} R^{a_3} \cdots = [a_0, a_1, a_2, \ldots]$

Image from Caroline Series' *The Geometry of Markoff Numbers*

Farey Tesselation

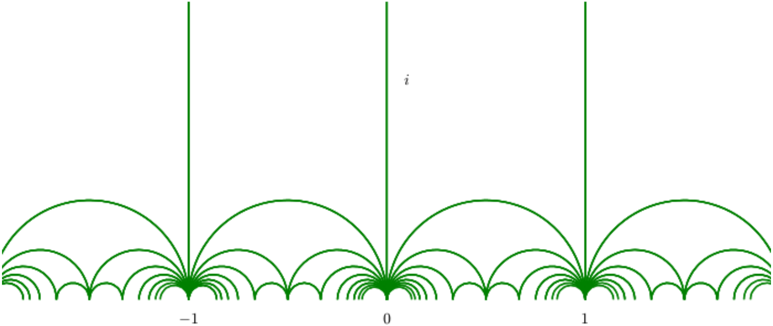
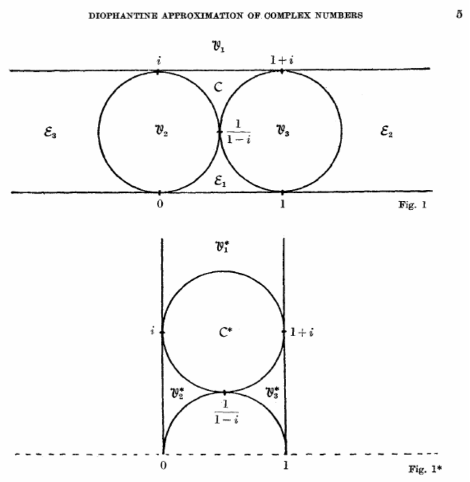


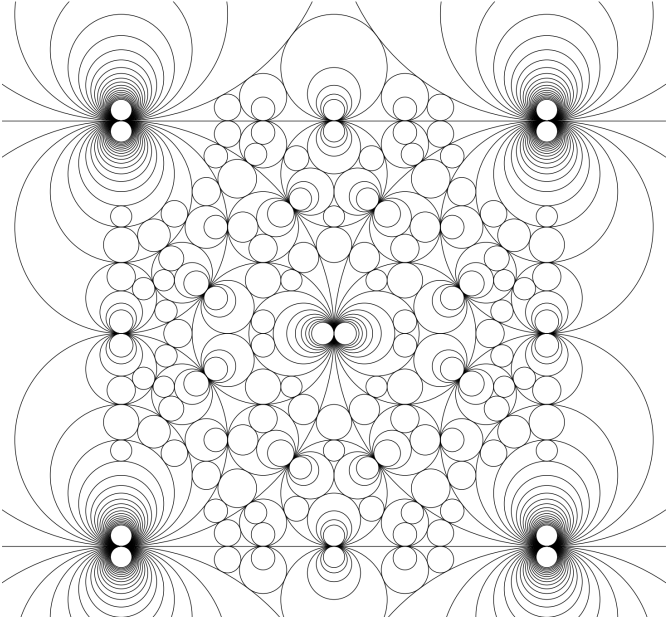
Image of $\{0, \infty\}$ (and its geodesic) under $\mathrm{PSL}_2(\mathbb{Z})$.

From Integers to Gaussian Integers

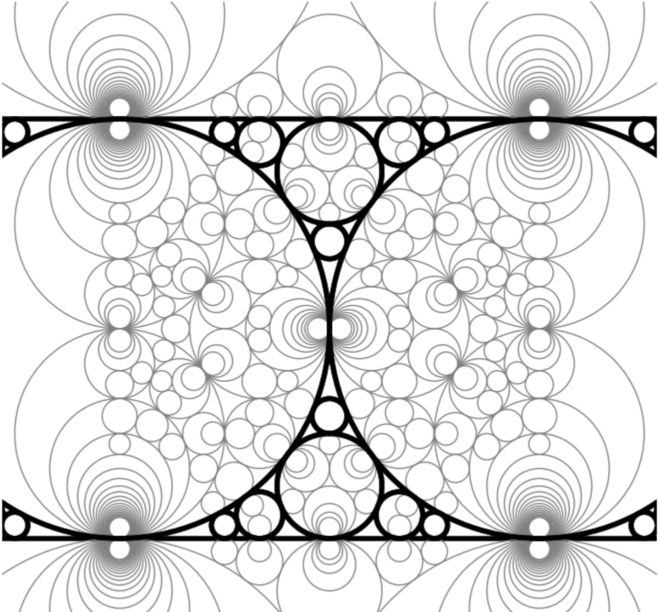


- Asmus Schmidt, *Diophantine Approximation of Complex Numbers*, Acta Arithmetica, 1975.
- Continued fractions for $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-2}]$ etc.

Orbit of $\widehat{\mathbb{R}}$ under $PSL_2(\mathbb{Z}[i])$



Orbit of $\widehat{\mathbb{R}}$ under $PSL_2(\mathbb{Z}[i])$



Schmidt arrangements of Quadratic Imaginary Fields

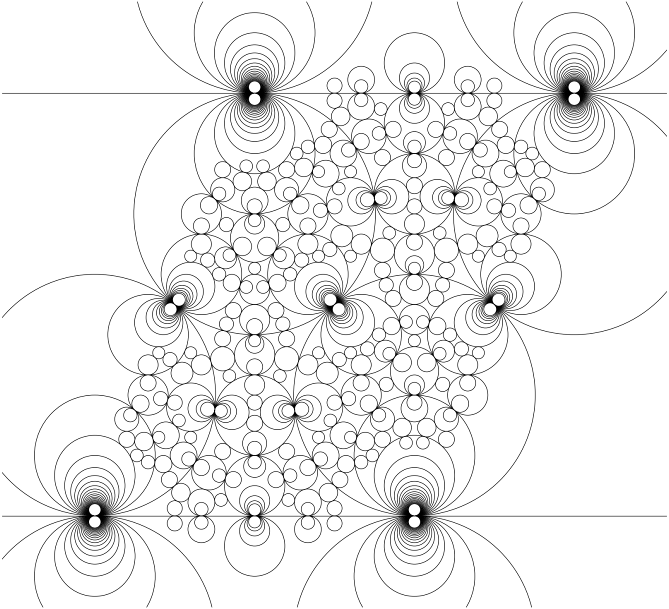
The *Schmidt arrangement* of a imaginary quadratic field K is the orbit of $\widehat{\mathbb{R}}$ under the Möbius transformations given by the *Bianchi group*

$$\mathrm{PSL}_2(\mathcal{O}_K) = \left\{ \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} : \alpha, \beta, \delta, \gamma \in \mathcal{O}_K,\ \alpha\delta - \beta\gamma = 1 \right\} / \pm I$$

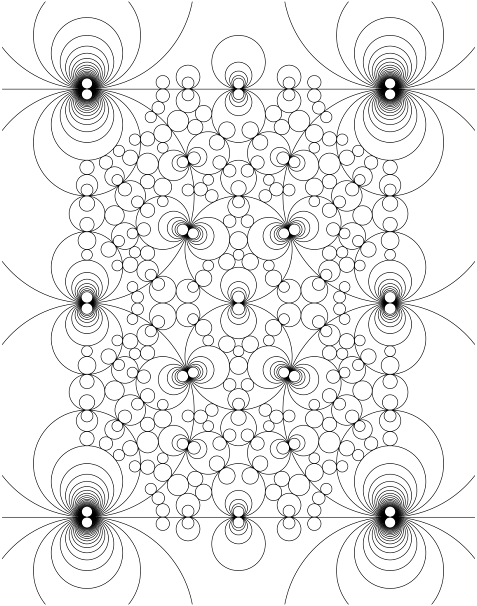
Each individual image $M(\widehat{\mathbb{R}})$ is called a *K-Bianchi circle*.

$$\mathcal{S}_K = \{K\text{-Bianchi circles}\}$$

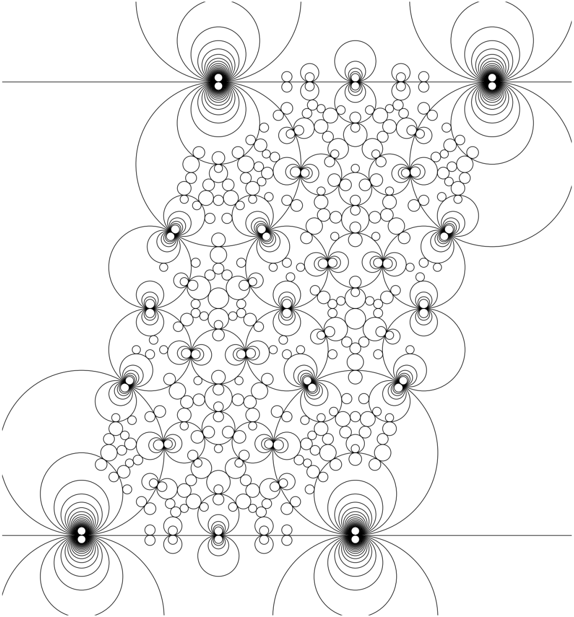
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-7})$



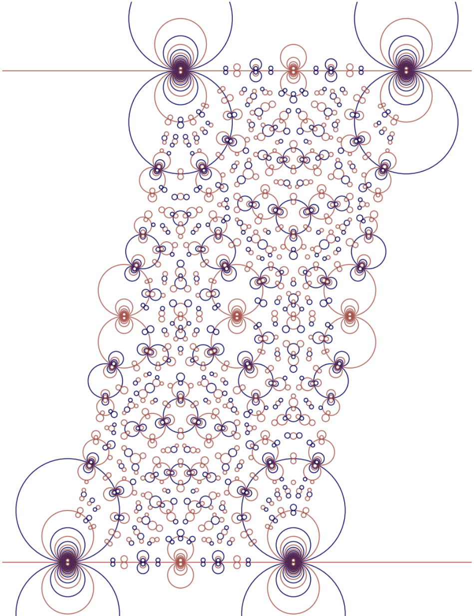
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-2})$



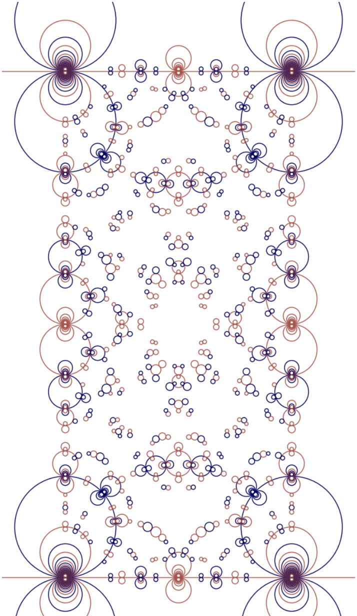
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-11})$



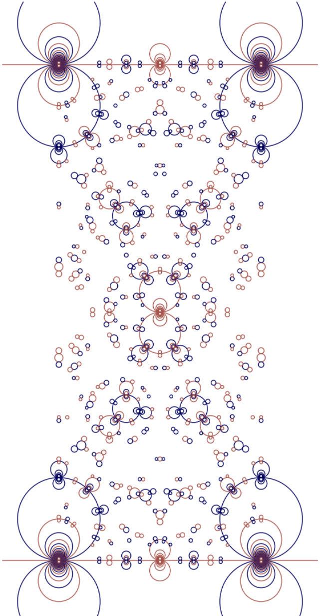
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-19})$



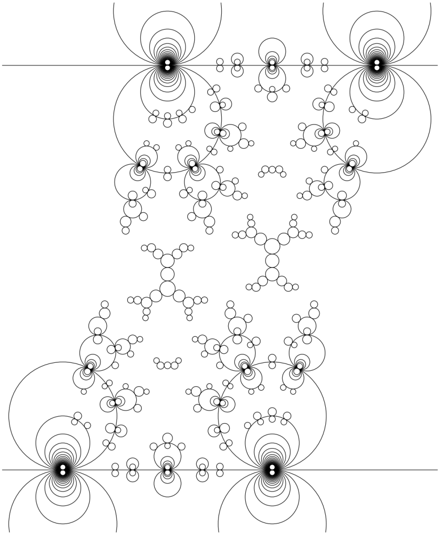
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-5})$



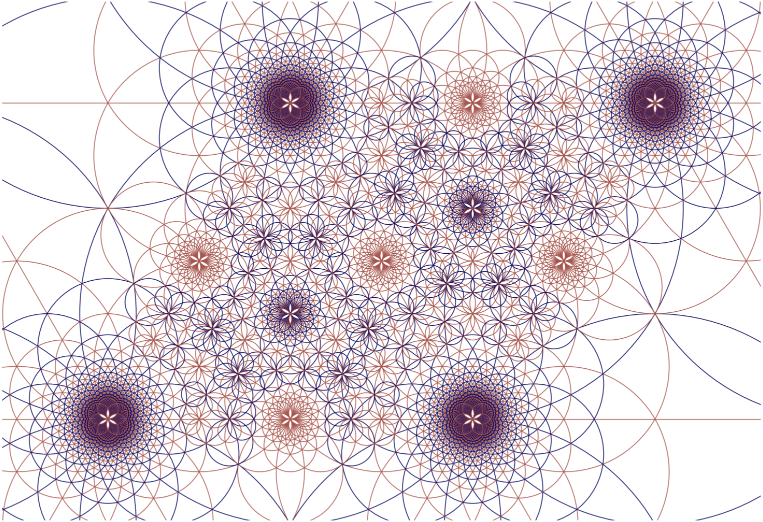
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-6})$



Schmidt Arrangement of $\mathbb{Q}(\sqrt{-15})$



Schmidt Arrangement of $\mathbb{Q}(\sqrt{-3})$



Basic properties of $\mathcal{S}_K$

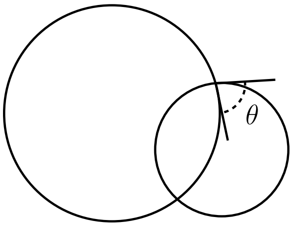
$$\Delta = \mathrm{Disc}(K)$$

Proposition (S.)

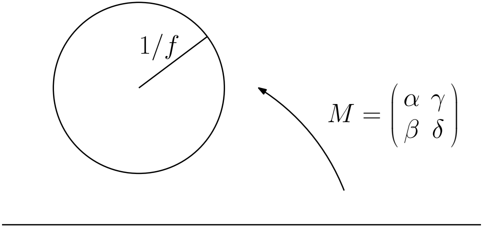
The curvatures in $\mathcal{S}_K$ lie in $\sqrt{-\Delta}\mathbb{Z}$.

Proposition (S.)

K-Bianchi circles intersect at points in K, at angles θ such that $e^{i\theta}$ is a unit in $\mathcal{O}_K$.



Circles are ideal classes



Theorem (S.)

$$\left\{ \begin{matrix} \textit{oriented} \\ \textit{circles} \end{matrix} \right\} \Big/ \left\{ \begin{matrix} \textit{translations by } \mathcal{O}_K \textit{ and} \\ \textit{rotations by `unit angles'} \end{matrix} \right\} \qquad M(\widehat{\mathbb{R}}) \qquad f = \textit{curvature}$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$\left\{ \underset{\mathfrak{a} \subset \mathcal{O}_f}{\begin{matrix} \textit{invertible} \\ \textit{ideal classes} \end{matrix}} \;\middle|\; f \in \mathbb{Z}^{>0}, \mathfrak{a}\mathcal{O}_K \sim \mathcal{O}_K \right\} \qquad \beta\mathbb{Z} + \delta\mathbb{Z} \qquad f = \textit{covolume}$$

Corollary: Number of circles of curvature f (up to equivalence) is h_f / h_K. (GLMWY for $\mathbb{Q}(i)$)

Euclideanity and $\mathcal{S}_K$

The *tangency graph* G_K of $\mathcal{S}_K$ is:

$$\left\{ \begin{array}{rcl} \textit{vertices} & = & \textit{circles} \\ \textit{edges} & = & \textit{tangencies} \end{array} \right\}.$$

Proposition (S.)

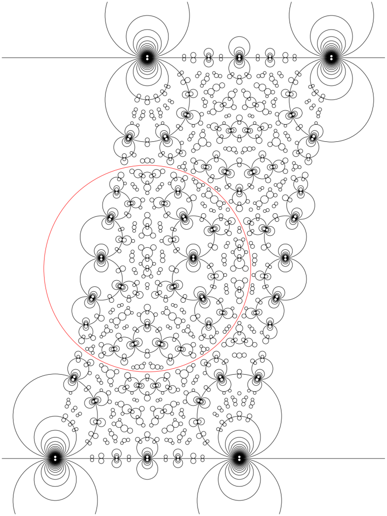
G_K is connected if and only if $\mathcal{O}_K$ is Euclidean.

Proof.

1. Connected component of $\widehat{\mathbb{R}}$ is all circles reachable by combinations of elementary matrices.
2. Thm of P.M. Cohn: $\mathcal{O}_K$ is Euclidean if and only if $\mathrm{SL}_2(\mathcal{O}_K)$ is generated by elementary matrices.

□

Euclideanity and $\mathcal{S}_K$



Theorem (S.)

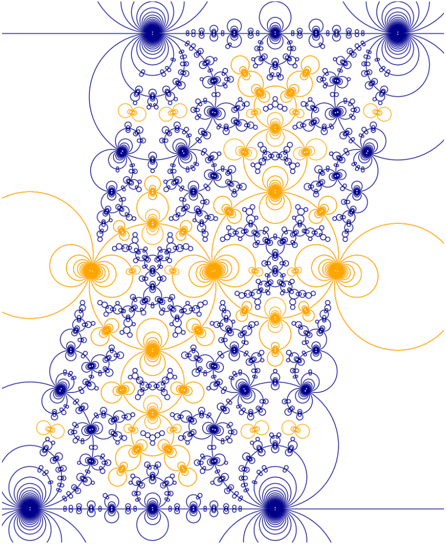
$\mathcal{S}_K$ is connected if and only if $\mathcal{O}_K$ is Euclidean.

The *ghost circle* is the circle orthogonal to the unit circle having center

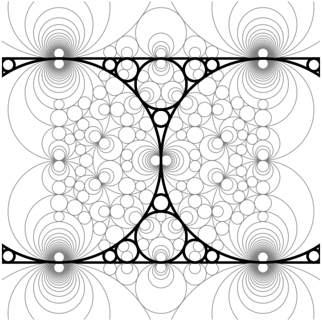
$$\begin{cases} \frac{1}{2} + \frac{\sqrt{\Delta}}{4} & \Delta \equiv 0 \pmod 4 \\ \frac{1}{2} + \frac{-\Delta-1}{4\sqrt{\Delta}} & \Delta \equiv 1 \pmod 4 \end{cases}.$$

It exists only when $\mathcal{O}_K$ is non-Euclidean.

Schmidt Arrangement of $\mathbb{Q}(\sqrt{-15})$ with Ghost Circles

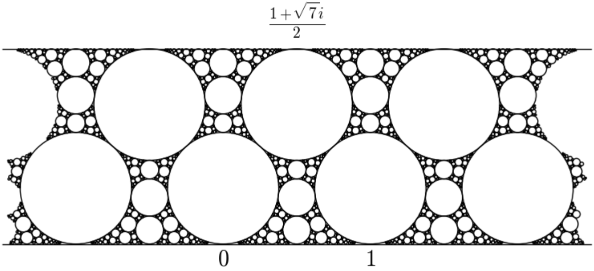


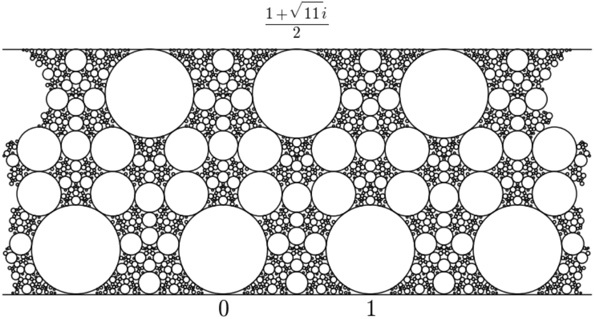
K-Apollonian Packings



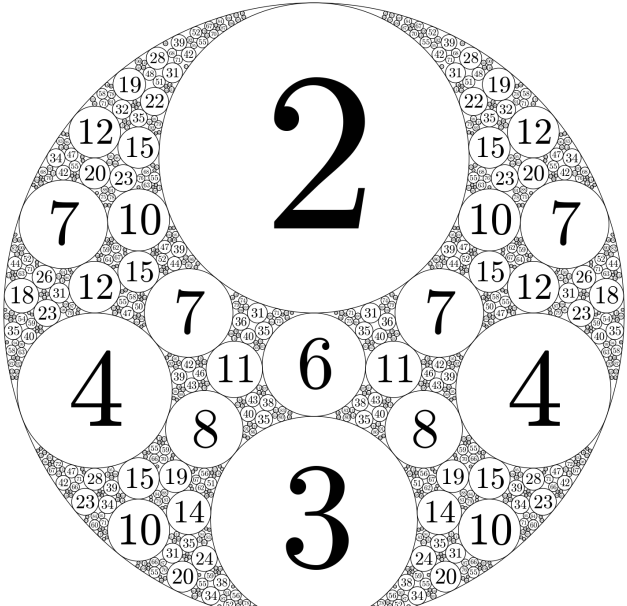
A K-Apollonian packing: $\mathcal{P}$ is a minimal non-empty set of circles that is closed under immediate tangency.

K-Apollonian Packings

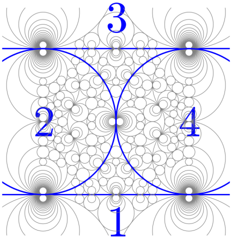


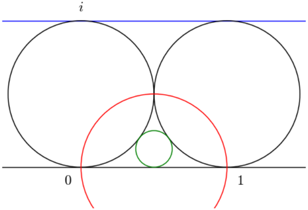


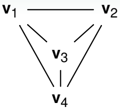
K-Apollonian Packings



Cheat Sheet for $\mathbb{Q}(i)$







$$W_D : \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$$

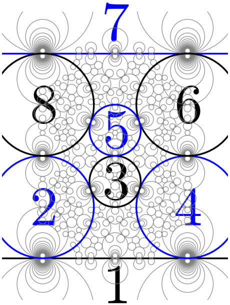
$$W_D^\dagger G_M W_D = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

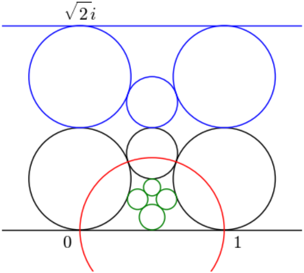
Apollonian group:

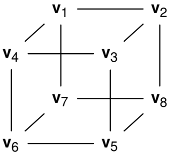
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$$\langle r, s, t, u : r^2 = s^2 = t^2 = u^2 = 1 \rangle$$

Cheat Sheet for $\mathbb{Q}(\sqrt{-2})$







$$W_D : \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6, \mathbf{v}_8$$

$$W_D^{\dagger} G_M W_D = \begin{pmatrix} 1 & -3 & -3 & -3 \\ -3 & 1 & -3 & -3 \\ -3 & -3 & 1 & -3 \\ -3 & -3 & -3 & 1 \end{pmatrix}$$

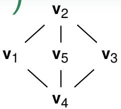
Apollonian group:

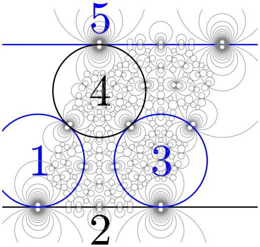
$$\begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 3 & 1 & 3 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 3 & 1 & 0 & 3 \\ 3 & 0 & 1 & 3 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

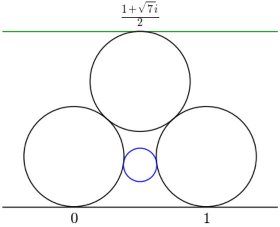
$$\begin{pmatrix} 1 & 3 & 3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 3 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 3 & 1 & 3 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & 0 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 3 & 3 & 0 & 1 \end{pmatrix}.$$

$$\langle r, s, t, u, v, w : r^2 = s^2 = t^2 = u^2 = v^2 = w^2 = 1 \rangle$$

Cheat Sheet for $\mathbb{Q}(\sqrt{-7})$







$W_D : \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$

$$W_D^\dagger G_M W_D =$$

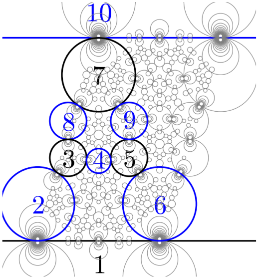
$$\begin{pmatrix} 1 & -1 & -5/2 & -1 \\ -1 & 1 & -1 & -5/2 \\ -5/2 & -1 & 1 & -1 \\ -1 & -5/2 & -1 & 1 \end{pmatrix}$$

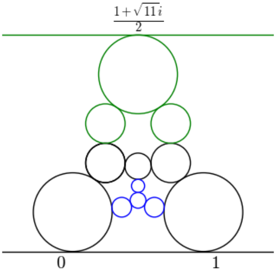
Apollonian group:

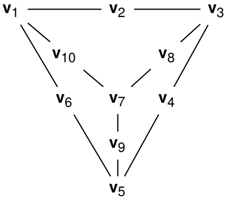
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 0 & -1 \\ 3 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 3 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 2 & 3 & 0 \end{pmatrix}.$$

$$\langle r, s, t : r^2 = s^2 = t^2 = 1 \rangle$$

Cheat Sheet for $\mathbb{Q}(\sqrt{-11})$







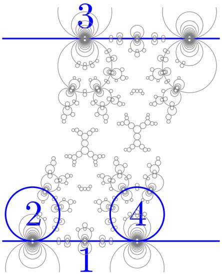
$W_D : \mathbf{v}_1 + \mathbf{v}_4, \mathbf{v}_1 + \mathbf{v}_8, \mathbf{v}_1 + \mathbf{v}_9, \mathbf{v}_3 + \mathbf{v}_9$

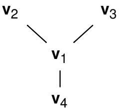
Apollonian group:

$$\begin{pmatrix} 1 & 3 & 3 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 3 & 1 & 3 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 3 & 3 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 3 & 3 & 3 & 1 \end{pmatrix}.$$

$$\langle r, s, t, u : r^2 = s^2 = t^2 = u^2 = 1 \rangle$$

Cheat Sheet for $\mathbb{Q}(\sqrt{\Delta})$, $\Delta < -11$





- Swaps: swap out $\mathbf{v}_i$ for $i = 2, 3, 4$ or move $\mathbf{v}_1$ to $\mathbf{v}_2$.

Apollonian group ($\Delta \equiv 0 \pmod 4$):

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1+\frac{\Delta}{4} & 1 & 1+\frac{\Delta}{4} \\ 0 & 1 & 0 & 0 \\ 1 & -\frac{\Delta}{4}-1 & 0 & -\frac{\Delta}{4}-1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Apollonian group ($\Delta \equiv 1 \pmod 4$):

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & -1 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \frac{\Delta+3}{4} & 0 \\ 1 & 1 & -\frac{\Delta-1}{4} & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -\frac{\Delta+3}{4} & 0 \end{pmatrix}.$$

$$\langle r, s, t, u : r^2 = s^2 = t^2 = u^2 = 1, rstu = stur \rangle$$

Generalized Local-Global Conjecture

Conjecture (S.)

$\mathcal{P}$ a primitive, integral K-ACP for $K \neq \mathbb{Q}(\sqrt{-3})$ with discriminant Δ. Let S_M be the set of residues of curvatures modulo M. Then, there exists an $M \mid 24$, so that any sufficiently large integer with a residue in S_M occurs as a curvature. A sufficient M is given by

$$v_2(M) = \begin{cases} 3 & \Delta \equiv 28 \pmod{32} \\ 2 & \Delta \equiv 8, 12, 20, 24 \pmod{32} \\ 1 & \Delta \equiv 0, 4, 16 \pmod{32} \\ 0 & \textit{otherwise} \end{cases},$$

$$v_3(M) = \begin{cases} 1 & \Delta \equiv 5, 8 \pmod{12} \\ 0 & \textit{otherwise} \end{cases}.$$

Apollonian generalizations

Guettler, Mallows, *A generalization of Apollonian packing of circles*, J. of Comb., 2010.

4 Gerhard Guettler and Colin Mallows

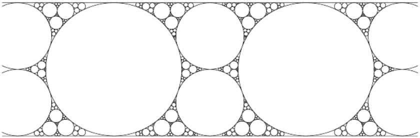
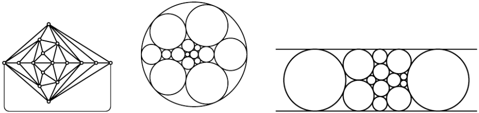


Figure 3: Another generalized Apollonian packing.

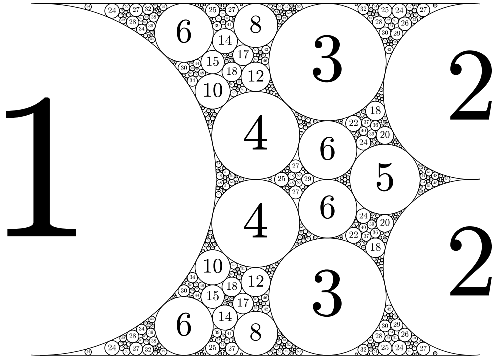
Butler, Graham, Guettler, Mallows, *Irreducible Apollonian Configurations and Packings*, Disc. Comp. Geom., 2010.



(a) Tangency graph (b) Generic packing (c) Standard packing

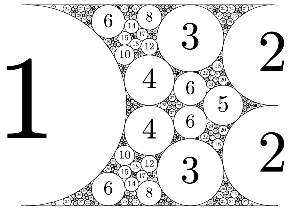
Fig. 2 Different representations of an Apollonian packing

Cuboctohedral Packing (Kontorovich-Nakamura)



Example lives in $M\ (PSL_2(\mathbb{Z}[\sqrt{-6}) \rtimes \mathfrak{c})\ M^{-1}$ where
$M = \begin{pmatrix} \sqrt{-6} & 0 \\ 0 & 1 \end{pmatrix}$.

A more general class of groups (with Fuchs, Zhang)



Let $\mathcal{A}$ be infinite-covolume, geometrically finite, Zariski dense Kleinian group (i.e. in $PSL_2(\mathbb{C})$), which is:

- not Fuchsian (not in $PSL_2(\mathbb{R})$)
- $PSL_2(\mathbb{Z}) \cap \mathcal{A}$ contains a principal congruence subgroup
- the entries of $\mathcal{A}$ are contained in a fractional ideal of a quadratic imaginary field

A more general class of groups (with Fuchs, Zhang)

Theorem (Fuchs-S.-Zhang)

Let $\mathcal{A}$ be a group as defined above. Let C be a circle tangent to $\widehat{\mathbb{R}}$, related to $\widehat{\mathbb{R}}$ by $PSL_2(K)$. Let $\mathcal{C}$ be the collection of curvatures appearing in the orbit $\mathcal{A} \cdot C$. Let $\mathcal{S}_M$ be the set of residues of $\mathcal{C}$ modulo M.
Then $\mathcal{C}$ contains density one of the integers whose residues modulo M are in $\mathcal{S}_M$.

Proof: Spectral gap (using geometry and Lie algebra), circle method as in Bourgain, Kontorovich.