On the inductive blockwise Alperin weight condition

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The blockwise Alperin weight conjecture

For a finite group G and a prime ℓ, an ℓ-weight means a pair (R, φ), where R is an ℓ-subgroup of G and $\varphi \in \mathrm{Irr}(N_G(R))$ with $R \leqslant \mathrm{Ker}\,\varphi$ of defect zero viewed as a character of $N_G(R)/R$.

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. **Conjecture (L. Alperin, 1986)** Let G be a finite group, ℓ a prime and B an ℓ-block of G, then $|\mathcal{W}(B)| = |\,\mathrm{IBr}(B)|$.

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Theorem (Späth, 2013)

Let G be a finite group and ℓ be a prime. Assume that every nonabelian simple group S involved in G satisfies the inductive BAW condition. Then the blockwise Alperin weight condition holds for every ℓ-block of G.

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- $\mathrm{PSL}_3(q)$ (Schulte, 2015, Z. Feng, C. Li, Z. Li, 2017).

Results for type A with cyclic outer automorphism groups

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- *If $n \geqslant 3$, $(n, q+1) = 1$ and $(n, q) \notin \{(4,2), (6,2)\}$, then the inductive BAW condition holds for every ℓ-block of $\mathrm{PSU}_n(q)$.*

Inductive BAW condition

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- (Partitions) There exist subsets $\mathrm{IBr}(B|Q) \subseteq \mathrm{IBr}(B)$ for every ℓ-radical subgroup Q of X with the following properties:
  - $\mathrm{IBr}(B|Q)^a = \mathrm{IBr}(B|Q^a)$ for every $Q \in \mathrm{Rad}_\ell(X)$ and $a \in \mathrm{Aut}(X)_B$,
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- (Bijections) For every $Q \in \mathrm{Rad}_\ell(X)$ there exists a bijection $\Omega_Q^X : \mathrm{IBr}(B|Q) \to \mathrm{dz}(N_X(Q)/Q, B)$ such that $\Omega_Q^X(\phi)^a = \Omega_{Q^a}^X(\phi^a)$ for every $\phi \in \mathrm{IBr}(B|Q)$ and $a \in \mathrm{Aut}(X)_B$.

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- (Normally Embedded Conditions).

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- (Normally Embedded Conditions).
- If B is of ℓ-defect zero, then $\Omega_{\{1\}}^X(\psi^\circ) = \psi$ for every $\psi \in \mathrm{Irr}(B)$, and $\tilde{\phi} = \tilde{\phi}'$ for every $\phi \in \mathrm{IBr}(B|\{1\})$.

Proof

Under our assumptions,

- The outer automorphism group of $X = \mathrm{SL}_n(\pm q) = \mathrm{PSL}_n(\pm q)$ is cyclic, then it suffices to prove the first two part of the inductive BAW condition, which means an $\mathrm{Aut}(X)$-equivariant bijection between irreducible Brauer characters and weights.

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- By the works of Alperin, Fong and An, we already have a bijection between irreducible Brauer characters and weights of $\mathrm{GL}_n(\pm q)$, then it suffices to consider the actions of automorphisms.

Proof: actions on ordinary characters

Jordan decomposition of characters: the irreducible characters of $\mathrm{GL}_n(\pm q)$ are in bijection with the $\mathrm{GL}_n(\pm q)$-conjugacy classes of pairs (s, μ), where s is a semisimple element of $\mathrm{GL}_n(\pm q)$ and $\mu = \prod_\Gamma \mu_\Gamma$ with $\mu_\Gamma \vdash m_\Gamma(s)$.

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Lemma

If χ is a character of $\mathrm{GL}_n(\pm q)$ corresponding to (s, μ) and σ is an automorphism of $\mathrm{GL}_n(\pm q)$, then χ^σ corresponds to $(\sigma(s), {}^\sigma\mu)$ where $({}^\sigma\mu)_{\sigma\Gamma} = \mu_\Gamma$.

Proof: actions on Brauer characters

- $\mathcal{E}(\mathrm{GL}_n(\pm q), \ell')$ is a basic set of $\mathrm{IBr}(G)$, where

$$\mathcal{E}(\mathrm{GL}_n(\pm q), \ell') = \bigcup_{s \in \mathbf{G}^F_{ss,\ell'}} \mathcal{E}(\mathrm{GL}_n(\pm q), s).$$

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- Since the decomposition matrix corresponding to $\mathcal{E}(\mathrm{GL}_n(\pm q), \ell')$ is unitriangular, there is an $\mathrm{Aut}(G)$-equivariant block-preserving bijection between $\mathcal{E}(\mathrm{GL}_n(\pm q), \ell')$ and $\mathrm{IBr}(G)$.

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- Then the actions of automorphisms on irreducible Brauer characters are just "permutations of elementary divisors".

Proof: actions on weights

- Let B be a block of $\mathrm{GL}_n(\pm q)$ with label (s, κ) (Fong, Srinivasan, Broué), then we can label all the B-weights by triples (s, κ, K), where $K = \prod_\Gamma K_\Gamma$ with K_Γ a collection of ℓ-cores.

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- Thus we can prove our theorem.

Definition of K

- Let (R, φ) be a weight, then $\varphi = \mathrm{Ind}_{N_G(R)_\theta}^{N_G(R)} \psi$, where $\theta \in \mathrm{Irr}(C_G(R)R)$ of defect zero as a character of $C_G(R)R/R$, $\psi \in \mathrm{Irr}(N_G(R)_\theta|\theta)$ of defect zero as a character of $N_G(R)_\theta/R$.

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- $R = R_0R_+$ and all constructions can be decomposed accordingly.
- $\theta_+ = \prod_{\Gamma,\delta,i} \theta_{\Gamma,\delta,i}^{t_{\Gamma,\delta,i}}, \quad R_+ = \prod_{\Gamma,\delta,i} R_{\Gamma,\delta,i}^{t_{\Gamma,\delta,i}}.$

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- $\theta_+ = \prod_{\Gamma,\delta,i} \theta_{\Gamma,\delta,i}^{t_{\Gamma,\delta,i}}, \quad R_+ = \prod_{\Gamma,\delta,i} R_{\Gamma,\delta,i}^{t_{\Gamma,\delta,i}}$.
- $N_+(\theta_+) = \prod_{\Gamma,\delta,i} N_{\Gamma,\delta,i}(\theta_{\Gamma,\delta,i}) \wr \mathfrak{S}(t_{\Gamma,\delta,i}), \quad \psi_+ = \prod_{\Gamma,\delta,i} \psi_{\Gamma,\delta,i}$, where

$$\psi_{\Gamma,\delta,i} = \mathrm{Ind}_{N_{\Gamma,\delta,i}(\theta_{\Gamma,\delta,i})\wr\prod_j \mathfrak{S}(t_{\Gamma,\delta,i,j})}^{N_{\Gamma,\delta,i}(\theta_{\Gamma,\delta,i})\wr\mathfrak{S}(t_{\Gamma,\delta,i})} \overline{\prod_j \psi_{\Gamma,\delta,i,j}^{t_{\Gamma,\delta,i,j}}} \cdot \prod_j \phi_{\kappa_{\Gamma,\delta,i,j}}$$

- Finally, $K_\Gamma : \quad \psi_{\Gamma,\delta,i,j} \mapsto \kappa_{\Gamma,\delta,i,j}$.

THANK YOU!