The Alperin Weight Conjecture for S_n and GL_n Revisited

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G a finite group, r a prime, B an r-block of G.

A weight of G is a pair (R, φ), where

- R is an r-subgroup of G,
- $\varphi \in (N_G(R)/R)_0^\vee$, i.e., φ is an irreducible character of $N_G(R)/R$ in an r-block of defect 0. So R is a radical r-subgroup of G, i.e., $R = O_r(N_G(R))$.

(R, φ) is a B-weight if the block b of $N_G(R)$ containing φ induces B in the sense of Brauer.

Let $\mathfrak{X}_B = \{$irreducible Brauer characters in $B\}$.

Let $\mathfrak{Y}_B = \{B$-weights of $G\}/\sim_G$.

The Alperin Weight Conjecture

$$|\mathfrak{X}_B| = |\mathfrak{Y}_B|$$

Alperin wrote in 1986 regarding the symmetric group S_n

"... the proof is an elaborate determination and count of the weights and that the result coincides without any apparent direct connection with the known results for the number of simple modules."

But in fact, $\mathfrak{X}_B$ and $\mathfrak{Y}_B$ for S_n are encoded by the same labels.

Nakayama: A block B of S_n is labeled by an r-core partition κ, where $n = |\kappa| + wr$ and the r-weight $w \geq 0$. By James we may view

$$\mathfrak{X}_B = \{\underset{\sim}{r\text{-regular partitions}}\ \lambda \text{ of } n \text{ with } r\text{-core } \kappa\},$$

i.e., $\lambda = 1^{m_1}2^{m_2}\cdots k^{m_k}$ with all $m_i < r$. For the characters in $\mathfrak{X}_B$ are characters of heads of certain Specht modules reduced modulo r, namely those labeled by r-regular partitions with r-core κ. Let

$$\mathfrak{X}'_B = \{(\lambda_1, \lambda_2, \ldots, \lambda_{r-1}) : \lambda_i \text{ partitions}, \ \textstyle\sum_i |\lambda_i| = w\}.$$

Then there are natural bijections

$$\mathfrak{X}_B \longleftrightarrow \mathfrak{X}'_B, \qquad \mathfrak{X}'_B \longleftrightarrow \mathfrak{Y}_B$$

$\mathfrak{X}_B \longleftrightarrow \mathfrak{X}'_B$ is due to Lascoux, Leclerc, and Thibon using work of Hayashi, Kac, Kashiwara, Kleshchev, Misra, and Miwa. Needed:

1) Kleshchev's connected r-good graph Γ_r

- Vertices are r-regular partitions of n for $n \geq 0$.
- Directed edges are colored by $I = \{0, 1, \ldots, r-1\}$. The edge $\lambda \longrightarrow_i \mu$ exists if adding a good i-node to λ gives μ.

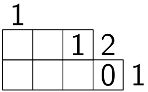
A node γ is an i-node for $i \in I$ if γ is in row s, column t, and $t - s \equiv i \pmod{r}$. Example: For $r = 3$

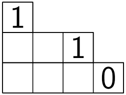
has i-nodes

1				
2	0	1	2	
0	1	2	0	1

Let λ be a Young diagram. A node $\gamma \in \lambda$ is removable if $\lambda \backslash \{\gamma\}$ is a Young diagram. A node $\gamma \notin \lambda$ is an addable node of λ if $\lambda \cup \{\gamma\}$ is a Young diagram.

Write the sequence of R's and A's for the removable and addable i-nodes occurring from left to right in λ. Remove any RA from the sequence; repeat until no RA remains. The first R and the last A in what remains are the good removable and addable i-nodes of λ.

For $r = 3$ adding the good 1-node to 

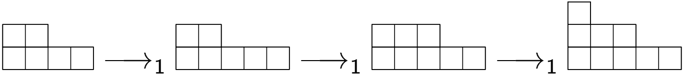
gives 

since the sequence of relevant 1-nodes is A(RA).

2) The action of the Weyl group W of $\widehat{\mathfrak{sl}}_r$ on vertices of Γ_r given by Kashiwara. This requires the Fock space $\mathcal{F}$ (the $U_q(\widehat{\mathfrak{sl}}_r)$-module $\bigoplus_\lambda \mathbb{Q}(q)\lambda$ with partitions λ as basis); the basic module $M(\Lambda_0)$ of $\mathcal{F}$; and a crystal basis for $M(\Lambda_0)$.

The actual effect of W on Γ_r: The fundamental reflection s_i of W reflects maximal strings of i-edges in Γ_r about their centers.

Example: Γ_3 has maximal 1-string



So $s_1 : (2,4) \longleftrightarrow (1,3,5), \quad (2,5) \longleftrightarrow (3,5).$

3) Let κ, κ' be r-cores in Γ_r. Then a result of Kac implies

- $w_\kappa(\emptyset) = \kappa$, $w_{\kappa'}(\emptyset) = \kappa'$ for some w_κ, $w_{\kappa'}$ in W.
- $w_{\kappa'} w_\kappa^{-1}$: $\{\lambda$ in Γ_r with r-core κ and r-weight $w\}$
  $\xrightarrow{\sim} \{\lambda'$ in Γ_r with r-core κ' and r-weight $w\}$
- The bijection is independent of the choice of w_κ and $w_{\kappa'}$.

Take $\kappa' = (1^{r-1}, 2^{r-1}, \ldots, k^{r-1})$ with $k \geq w - 1$. The form of κ' implies the partitions λ' in the second set have r-quotients $(\lambda'_0, \lambda'_1, \ldots, \lambda'_{r-1})$ with $\lambda'_j = \emptyset$ for $j \equiv k \pmod r$ using abacus diagrams with mr beads for the r-quotients. So $w_{\kappa'} w_\kappa^{-1}$ induces

$$\mathfrak{X}_B \longrightarrow \mathfrak{X}'_B$$

$\mathfrak{X}'_B \longrightarrow \mathfrak{Y}_B$ requires the *r-core tower* of a partition λ.

- Let λ have r-core λ^0 and r-quotient $(\lambda_0, \lambda_1, \ldots, \lambda_{r-1})$.
- Let λ_i have r-core λ_i^0 and r-quotient $(\lambda_{i0}, \lambda_{i1}, \ldots, \lambda_{i,r-1})$.
- Let λ_{ij} have r-core λ_{ij}^0 and r-quotient $(\lambda_{ij0}, \lambda_{ij1}, \ldots, \lambda_{ij,r-1})$.

Let $I = \{0, 1, \ldots, r-1\}$. The r-core tower of λ is

$$\{\lambda_{\bar{u}}^0 : \bar{u} \in I^h,\ h \geq 0\}.$$

Then $|\lambda| = \sum_{h \geq 0} \sum_{\bar{u} \in I^h} r^h |\lambda_{\bar{u}}^0|$.

Fix a Sylow r-subgroup A of S_r. A basic r-group of S_{r^ℓ} has form

$$A_k = A * A * \cdots * A \quad (\ell \text{ factors}, \ \ell \geq 1),$$

where $* \in \{\otimes, \wr\}$ and $\otimes$'s are performed before $\wr$'s.

The signature $\sigma(A_k)$ is the $(\ell - 1)$-tuple of $\otimes$'s and $\wr$'s defining A_k and characterizes A_k up to conjugacy in S_{r^ℓ}. The length of A_k is ℓ; the depth of A_k is one plus the number of $\wr$'s in $\sigma(A_k)$.

Example: $A \otimes A \wr A \wr A \otimes A \otimes A$ has signature $(\otimes, \wr, \wr, \otimes, \otimes)$, length 6, and depth 3.

Theorem: Let (R, φ) be a weight of S_n. Then

$$\begin{aligned} n &= n_0 + n_1 + \cdots + n_s \\ R &= R_0 \times R_1 \times \cdots \times R_s \end{aligned}$$

- R_0 is the 1-subgroup of S_{n_0}
- R_i is a basic subgroup of S_{n_i} for $i \geq 1$.

To describe φ in $(N_{S_n}(R)/R)_0^\vee$ write

$$R = R_0 \times \prod_k {A_k}^{\Omega_k}$$

where the A_k are different basic subgroups.

Let N_k be the normalizer of A_k in its ambient symmetric group.

$$\begin{aligned} N_{S_n}(R)/R &= S_{n_0} \times \prod_k (N_k/A_k) \wr S(\Omega_k) \\ \varphi &= \varphi_0 \times \prod_k \varphi_k, \end{aligned}$$

- $\varphi_0 \in (S_{n_0})_0^\vee$ has label an r-core partition of n_0.
- $\varphi_k \in ((N_k/A_k) \wr S(\Omega_k))_0^\vee$. So φ_k is given by an assignment

$$f_k \colon (N_k/A_k)_0^\vee \to \{r\text{-cores}\}, \quad \sum_\psi |f_k(\psi)| = |\Omega_k|$$

by the character theory of wreath products.

Namely, partition $\Omega_k = \coprod_{\psi \in (N_k/A_k)_0^\vee} \Omega_{k\psi}$ with $|\Omega_{k\psi}| = |f_k(\psi)|$.

The assignment f_k gives

- $\prod_\psi \psi^{\Omega_{k\psi}}$, a character of the base group $(N_k/A_k)^{\Omega_k}$ which extends canonically to its stabilizer T in $(N_k/A_k) \wr S(\Omega_k)$.
- $\prod_\psi \chi_{f_k(\psi)}$, a character of $T/(N_k/A_k)^{\Omega_k} \simeq \prod_\psi S(\Omega_{k\psi})$.

Inducing the product of the two characters of T to $(N_k/A_k) \wr S(\Omega_k)$ then gives φ_k.

And $(N_k/A_k)_0^\vee$? If A_k has depth t and the numbers of $\otimes$'s between successive $\wr$'s in $\sigma(A_k)$ are $c_1 - 1,\ c_2 - 1, \ldots, c_t - 1$, then

$$N_k/A_k = \mathrm{GL}(c_1, r) \times \mathrm{GL}(c_2, r) \times \cdots \times \mathrm{GL}(c_t, r).$$

Example: If $\sigma(A_k) = (\otimes, \wr, \wr, \otimes, \otimes)$, then

$$N_k/A_k = \mathrm{GL}(2, r) \times \mathrm{GL}(1, r) \times \mathrm{GL}(3, r).$$

This is because

$$N(A \otimes \cdots \otimes A)/(A \otimes \cdots \otimes A) \simeq \mathrm{GL}(c, r) \quad (c \text{ factors } A),$$

$$N(X \wr Y)/(X \wr Y) \simeq N(X)/X \times N(Y)/Y,$$

where normalizers are in the appropriate ambient symmetric groups.

Thus

$$(N_k/A_k)_0^\vee = \prod_{i=1}^{t} \mathrm{GL}(c_i, r)_0^\vee.$$

Now

$$\mathrm{GL}(c, r)_0^\vee = \{\mathrm{St}_1, \mathrm{St}_2, \ldots, \mathrm{St}_{r-1}\},$$

where $\mathrm{St}_i = \mathrm{St} \times (\xi^i \circ \det)$, St is the Steinberg character, and ξ generates $(\mathbf{F}_r^\times)^\vee$. Let $I_+ = \{1, 2, \ldots, r-1\}$. So I_+^t labels the characters in $(N_k/A_k)_0^\vee$ and we may view the assignment f_k of φ_k as an assignment

$$f_k \colon I_+^t \to \{r\text{-cores}\}, \qquad \sum_{\bar{v} \in I_+^t} |f_k(\bar{v})| = |\Omega_k|.$$

To define the bijection $\mathfrak{X}'_B \to \mathfrak{Y}_B$, $\Lambda \mapsto (R_\Lambda, \varphi_\Lambda)$, where

$$R_\Lambda = R_0 \times \prod_k A_k^{\Omega_k}, \quad \varphi_\Lambda = \varphi_0 \times \prod_k \varphi_k,$$

we need

- The $|\Omega_k|$'s.
- The assignments f_k's defining the φ_k's.
- $n_0 = |\kappa|$ so φ_0 can be labeled by κ.

Let $\Lambda = (\lambda_1, \ldots, \lambda_{r-1}) \in \mathfrak{X}'_B$ and let $\{\lambda^0_{i;\bar{u}} : \bar{u} \in I^h,\ h \geq 0\}$ be the r-core tower of λ_i. The signature $\sigma(\bar{u})$ of $\bar{u} \in I^h$ is the tuple gotten from $\bar{u}$ by replacing zeros by $\otimes$ and non-zeros by $\wr$.

Suppose A_k has length ℓ and depth t.

- Take $|\Omega_k| = \sum_{i=1}^{r-1} \sum_{\substack{\bar{u} \in I^{\ell-1} \\ \sigma(\bar{u}) = \sigma(A_k)}} |\lambda^0_{i;\bar{u}}|$.

If $\sigma(\bar{u}) = \sigma(A_k)$, then $\bar{u}$ has $t-1$ non-zero entries.

Let $\bar{u}_+$ be the $(t-1)$-tuple of these non-zero entries.

- Take $f_k : (i; \bar{u}_+) \mapsto \lambda^0_{i;\bar{u}}$ for $\bar{u} \in I^{\ell-1}$, $\sigma(\bar{u}) = \sigma(A_k)$.
- Take $\varphi_0 = \chi_\kappa$. For $n_0 = |\kappa|$ since $\sum_{i=1}^{r-1} |\lambda_i| = w$.

Then $(R_\Lambda, \varphi_\Lambda)$ is a weight. $(R_\Lambda, \varphi_\Lambda)$ is even a B-weight by a result of Marichal-Puig. This gives $\mathfrak{X}'_B \longrightarrow \mathfrak{Y}_B$.

Let $r = 3$. Let B be the block of $G = S_6$ labeled by $\kappa = \emptyset$. Let B' be the block labeled of $G' = S_8$ labeled by by $\kappa' = (1^2)$.

(6)	$(1,7)$	$((2),\emptyset,\emptyset)$	$\psi_1 \mapsto (2),\ \psi_2 \mapsto \emptyset$
$(1^2,4)$	$(2^2,4)$	$((1^2),\emptyset,\emptyset)$	$\psi_1 \mapsto (1^2),\ \psi_2 \mapsto \emptyset$
$(1,2,3)$	$(1,2^2,3)$	$(\emptyset,\emptyset,(1^2))$	$\psi_1 \mapsto \emptyset,\ \psi_2 \mapsto (1^2)$
$(1,5)$	$(2,6)$	$(\emptyset,\emptyset,(2))$	$\psi_1 \mapsto \emptyset,\ \psi_2 \mapsto (2)$
(3^2)	(4^2)	$((1),\emptyset,(1))$	$\psi_1 \mapsto (1),\ \psi_2 \mapsto (1)$

B-weights have the form (A^2, φ) since A is the only basic subgroup of G. φ is given by an assignment $f\colon (N/A)_0^\vee \to \{3\text{-cores}\}$ and $(N/A)_0^\vee = \mathrm{GL}(1,3)_0^\vee = \{\psi_1, \psi_2\}$ with $\psi_2 = 1$.

Let $G = \mathrm{GL}(n, q)$ and let

$$\mathcal{F} = \{\text{monic, irreducible polynomials } \Gamma \text{ in } \mathbf{F}_q[x]\}.$$

The $\chi_{s,\lambda}$ in $G^\vee$ have Jordan labels (s, λ), where

- s is a semisimple element of G determined up to G-conjugacy.
  $s = \prod_{\Gamma \in \mathcal{F}} s_\Gamma$, where s_Γ is the Γ-primary component of s.
- $\lambda = \prod_{\Gamma \in \mathcal{F}} \lambda_\Gamma$, where $\lambda_\Gamma \vdash m_\Gamma(s)$, the multiplicity of Γ in s.

Let B be an r-block of G, where $r \neq 2$ and $(r, q) = 1$. The Brauer characters in $\mathfrak{X}_B$ are not known, but $\sum_{\varphi \in \mathfrak{X}_B} \mathbf{Z}\varphi$ has a known basis.

For Γ in $\mathcal{F}$ let

- d_Γ be the degree of Γ,
- e_Γ be the multiplicative order of q^{d_Γ} modulo r.

B has a Jordan label (s, κ), where

- s is a semisimple r'-element determined up to G-conjugacy.
- $\kappa = \prod_{\Gamma \in \mathcal{F}} \kappa_\Gamma$, and κ_Γ is the e_Γ-core of a partition of $m_\Gamma(s)$.

$\sum_{\varphi \in \mathfrak{X}_B} \mathbf{Z}\varphi$ has basis

$$\mathfrak{X}'_B = \{\chi_{s,\lambda} \in G^\vee : \lambda_\Gamma \text{ has } e_\Gamma\text{-core } \kappa_\Gamma \text{ for } \Gamma \in \mathcal{F}\}$$

The $\chi_{s,\lambda}$ in $\mathfrak{X}'_B$ define B-weights (R,φ) in the manner for S_n with additional elaboration.

- $R = R_0 \times \prod_k {A_k}^{\Omega_k}$, where basic subgroups $A_k = Z_d \otimes E_\gamma \wr A_{\bar{s}}$ are composed of a cyclic $Z_d > 1$, an extra-special E_γ of order $r^{2\gamma+1}$ and exponent r, and a basic subgroup $A_{\bar{s}}$ of S_n with signature $\bar{s}$. Here $Z_d = O_r(\mathrm{GL}(1, q^{de}))$, where e is the order of q^d modulo r.
- Each ψ in $(N_k/A_k)_0^\vee$ has a well-defined type Γ in $\mathcal{F}$. So

$$(N_k/A_k)_0^\vee = \coprod_{\Gamma\in\mathcal{F}} (N_k/A_k)_{0\Gamma}^\vee$$

  where $(N_k/A_k)_{0\Gamma}^\vee$ is the subset of ψ's of type Γ.

The type of $\psi \in (N_k/A_k)_0^\vee$ is gotten as follows:

- Fix a $\theta \in (C_k A_k/A_k)_0^\vee$ in $\psi|_{C_k A_k/A_k}$, where $C_k = C_{G_k}(A_k)$.
- $\theta|_{C_k}$ is trivial on $Z(A_k) = C_k \cap A_k$. So $\theta|_{C_k}$ is the canonical character of a block β of C_k with defect group $Z(A_k)$.
- $A_k = Z_d \otimes E_\gamma \wr A_{\bar{s}}$ implies $C_k \simeq \mathrm{GL}(u, q^v)$, where r divides $q^v - 1$. So β has Jordan label $(s, -)$ with $s \in \mathrm{GL}(u, q^v)$.
- In the ambient $\mathrm{GL}(uv, q)$ containing $\mathrm{GL}(u, q^v)$, s has primary decomposition Γ^{e_Γ} for $\Gamma \in \mathcal{F}$. This Γ is the type of ψ.

The type allows a Jordan decomposition $\prod_{\Gamma \in \mathcal{F}} (R_\Gamma, \varphi_\Gamma)$ of (R, φ):

- $(R_\Gamma, \varphi_\Gamma)$ is a B_Γ-weight of $\mathrm{GL}(n_\Gamma, q)$, where B_Γ has label $(s_\Gamma, \kappa_\Gamma)$. Basic groups A_k in R_Γ have parameters $(d_\Gamma, \gamma, \bar{s})$. $\sigma(A_k) = (\otimes, \otimes, \ldots, \otimes)$, $(\otimes, \otimes, \ldots, \otimes, \wr)$, or $(\otimes, \otimes, \ldots, \otimes, \wr) \cup \bar{s}$.
- Assignments f_k for φ_k in φ_Γ have support in $(N_k/A_k)^\vee_{0\Gamma}$.
- $(N_k/A_k)^\vee_{0\Gamma} \simeq [1, e_\Gamma] \times I_+^t$, where t is the depth of $A_{\bar{s}}$.
- Suppose $\chi_{s_\Gamma, \lambda_\Gamma} \in \mathfrak{X}'_{B_\Gamma}$. The r-core towers $\{\lambda^0_{\Gamma i;\bar{u}}\}$ of the e_Γ-quotient $\{\lambda_{\Gamma 1}, \lambda_{\Gamma 2}, \ldots, \lambda_{\Gamma e_\Gamma}\}$ of λ_Γ define a B_Γ-weight as in the S_n case.