

# New Perspectives in Representation Theory of Finite Groups

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## 1 Overview of the Field

Group Theory is essentially the theory of symmetry for mathematical and physical systems, with major impact in diverse areas of mathematics. The Representation Theory of Finite Groups is a central area of Group Theory, with many fascinating and deep open problems, and significant recent successes. In 1963 R. Brauer [B] formulated a list of deep conjectures about ordinary and modular representations of finite groups. These have led to many new concepts and methods, but basically all of his main conjectures are still unsolved to the present day. A new wealth of difficult problems, relating global and local properties of finite groups, was opened up in the seventies and eighties (of the 20th century) by subsequent conjectures of J. McKay [Mc], J. Alperin [Al], E. Dade [D1, D2], M. Broué [Br], and others, all remain open up to date.

The classification of finite simple groups raised the hope that one should be able to reduce some of the aforementioned conjectures to statements about simple groups, and subsequently establish these statements by exploring deep knowledge about simple groups provided by the Deligne–Lusztig theory and other recent fundamental results in representation theory. In fact, this reduction to simple groups was accomplished by several of the organizers in [IMN] for the McKay conjecture, in [NT1] for the Alperin weight conjecture and in [S1], [S2] and [S3] for the blocks versions. This hope has recently materialized since the proof of the McKay conjecture for the prime 2 by two of our organizers: G. Malle and B. Späth [MS]. The road to prove the McKay conjecture for odd primes is now paved.

## 2 Recent Developments and Open Problems

There were two main themes in our meeting, with several remarkable highlights, which are discussed in more detail below. One of the directions that became even more apparent is the advent of powerful geometric methods to solve problems in finite group representation theory. This was witnessed for example in the impressive talks by Raphaël Rouquier and by Cedric Bonnafé. One of the most important problems in Representation Theory is to find Brauer’s decomposition matrices of symmetric groups and finite groups of Lie type. R. Rouquier announced an astonishing conjecture, joint work with O. Dudas, that will have a lasting impact on our field. The other theme has been remarkable progress on the so called McKay–Galois conjecture, proposed in 1994 by one of the organizers ([N]). Other themes are related to another open problem: how the absolute Galois group act on the irreducible characters of the simple groups.

## 2.1 Decomposition Numbers

Decomposition numbers are the data relating the modular representations of a finite group  $G$  over a field of positive characteristic to the generally better known complex characters  $\text{Irr}(G)$ . The two representation theories (modular and over the complex numbers) are very different in nature. Complex characters were introduced by Frobenius at the end of the 19th century and from the start corresponding sets  $\text{Irr}(G)$  were quickly determined for many of the finite simple groups known at the time, as for example for the symmetric and alternating groups. Modular representation theory was only introduced much later by Brauer in the 1940's and very simple questions about the representations of the symmetric groups are still unanswered — and intensely investigated, including in the present meeting.

A related question is the one of basic sets, that are subsets of  $\text{Irr}(G)$  somehow standing for the much more mysterious irreducible modules over fields of positive characteristic.

Decomposition numbers and basic sets are thus a vital information to tackle conjectures about modular representations since their versions over the complex numbers are generally much simpler. Any progress in the knowledge of decomposition numbers for finite simple groups, i.e., now essentially groups of Lie type, is of foremost importance.

It was a big success of this meeting to arrange the presence of Raphaël Rouquier (UCLA) one of the main contributors to the field of representation theory as a whole, and a very fruitful presence in our meeting as testified by numerous participants (see testimonials below). The meeting was the place where it was announced what is probably a breakthrough in the knowledge on decomposition numbers. In joint work with Olivier Dudas (Paris), Rouquier gives a conjectural description of the decomposition matrix of finite unitary groups with respect to the so-called non linear (non-defining) primes. This is the main case left open by the first generation of work on the subject (Dipper–Gruber–Hiss, 1985–1997) treating essentially linear primes [Di], [GH]. This has been confirmed for dimensions up to  $n$  by comparison with the known results. Though still a conjecture this description has the advantage of being very precise and concrete. The matrix should correspond to an explicit operator  $\nabla$  on the algebra of symmetric functions defined in terms of Macdonald polynomials.

Rouquier adds even more perspective to the subject by relating this algebraic setup to algebraic geometry. Indeed he provides a categorification of the operator  $\nabla$  in the category of coherent sheaves on Hilbert schemes on the complex plane. This interpretation should also open the door to (conjectural) generalizations to other types of classical groups for non linear primes.

## 2.2 The McKay Conjecture and refinements

If  $G$  is a finite group, we denote by  $\text{Irr}(G)$  the set of irreducible complex characters of  $G$  and by  $\text{Irr}_{p'}(G)$  its subset consisting of characters having degree not divisible by  $p$ . The McKay conjecture [Mc] from 1972 is at the center of representation theory. If  $p$  is a prime and  $G$  is a finite group, then it asserts that

$$|\text{Irr}_{p'}(G)| = |\text{Irr}_{p'}(\mathbf{N}_G(P))|,$$

where  $P$  is a Sylow  $p$ -subgroup of  $G$  and  $\mathbf{N}_G(P)$  its normalizer. That is to say, a certain fundamental information on  $G$  is encoded in some local subgroup of  $G$ , namely the Sylow normalizer  $\mathbf{N}_G(P)$ .

This conjecture somehow gave a new direction to representation theory of finite groups. Many results and conjectures, even recently, would not have been possible without it. We have already mentioned the conjectures by Alperin–McKay, Alperin, Dade and Broué. In fact major contributions to the representation theory of finite groups of Lie type such as [BMM] originate in the same kind of ideas.

Since our previous meeting in 2014, there has been an absolute highlight in this subject: Malle and Späth (two of our organizers) proved the McKay conjecture for  $p = 2$  (see [MS]), the case that was originally conjectured. This came after the reduction theorem by I. M. Isaacs, G. Malle and G. Navarro in [IMN], proving that the path that started in 2007 is a successful one. Of course, now the big question is what happens if  $p$  is odd? In our meeting we had a fruitful session on a discussion on what remains to do next, and on several possible ideas to tackle this long-standing problem.

In 2004, Navarro proposed a deep refinement of the McKay conjecture [N] that takes into account fields of values over the  $p$ -adics. This conjecture has consequences in determining local structure of the group from global structure and vice versa. In our meeting, A. Schaeffer Fry announced a proof, by her and in part also in

collaboration with J. Taylor, of the self-normalizing 2-Sylow conjecture, which is one of the most remarkable consequences of this McKay-Galois conjecture. This proof comes after 14 years since the conjecture was announced (and ten years after the proof [NTT] of the self-normalizing Sylow conjecture for odd primes). Recent work of Schaeffer Fry and Taylor also led to a proof of another conjecture of Navarro, Tiep, and Vallejo [NTV] concerning groups whose Sylow 2-normalizers contain a single irreducible 2-Brauer character (which was recently reduced to simple groups by Navarro and Vallejo). O. Brunat announced a proof of the McKay-Galois conjecture for symmetric groups and odd primes (the case  $p = 2$  was announced in our 2014 meeting) and Lucas Ruhstorfer announced a proof of the conjecture for groups of Lie type in their defining characteristic.

In recent years, there has been considerable interest in fields of values of characters, and a main open problem in this area is to determine how the absolute Galois group acts on the irreducible complex characters of a group. D. Rossi announced a proof of one direction of the 3-rational conjecture (by two of the organizers, Navarro and Tiep) that asserts that the finite groups with 3 rational conjugacy classes are the finite groups with 3 rational characters. For the other direction, he was also able to get a very tight control over a minimal nonsolvable counterexample (if any).

J. Tent announced his  $\mathbb{Q}_p$ -conjecture, that would extend work of J. Thompson, W. Feit and R. Gow. Almost since the beginning of character theory, the connections between fields and characters have been studied. An object of interest is the so called rational groups, or groups all of whose characters have values in the field of the rationals. While the non-abelian composition factors of rational groups were classified by W. Feit and G. Seitz long time ago, it is a conjecture of J. Thompson that the abelian composition factors are cyclic groups of prime order  $p$ , for  $p < 7$ . The conjecture is still open. Tent conjectures that if the field of values of the characters are contained in the field  $\mathbb{Q}_p$ , then the abelian composition factors of  $G$  are also under absolute control. Perhaps, further relations hold for arbitrary fields.

### 2.3 Future directions

Recent progress on all these fundamental conjectures raises the hope that complete proofs of some of them may be possible in the not too distant future.

It has become even clearer in the past few years that any further significant progress can only be achieved once we can resolve a number of basic questions on representations of finite groups of Lie type. With many experts in the Deligne-Lusztig theory present at the meeting, we had a discussion session on Tuesday afternoon to draw a roadmap towards possible solutions of some of these principal obstacles.

## 3 Presentation Highlights

We already highlighted the spectacular announcement by Raphaël Rouquier of his and Dudas' conjecture on the decomposition numbers of the finite unitary groups, as well as the impressive talk of Cedric Bonnafé presenting his results, joint with P. Shan and R. Maksimau, that established a conjecture concerning equivariant cohomology and fixed points of Calogero-Moser spaces in the smooth case.

Jesper Grodal presented in Banff a novel approach to endo-trivial modules. These representations of finite groups over fields of positive characteristic  $p$  were introduced by Dade in his study of the Glauberman correspondence. While modules of a given finite group  $G$  over a field  $k$  of characteristic  $p$  have been shown to be in general beyond classification, the subset of endo-trivial has been shown to be endowed with the structure of a finitely generated abelian group  $T_k(G)$  for the tensor product. This also relates with the Picard group of the stable category for the corresponding group algebra  $kG$ . Results by Alperin and Bouc-Carlson-Thévenaz (2001–2006, see [CT1]) have given a complete structure theorem in the case where  $G$  is a  $p$ -group.

In the general case of a finite group with Sylow  $p$ -subgroup  $S$ , one classically defines a restriction morphism

$$T_k(G) \rightarrow T_k(S)$$

whose kernel  $T_k(G, S)$  contains in fact all the relevant information to deduce  $T_k(G)$  from the case of  $p$ -groups thanks to work of Carlson-Thévenaz-Mazza-Nakano giving the image and proving the splitness of the above map. Grodal presented spectacular theorems about  $T_k(G, S)$ , identifying it with the first cohomology group of the orbit category  $\mathcal{O}$  on non-trivial  $p$ -subgroups with values in the units  $k^\times$ , viewed as a constant

coefficient system. He uses homotopical techniques to give a number of formulas for  $T_k(G, S)$  in terms of one-dimensional representations of normalizers and centralizers. This allows him to prove a conjecture by Carlson–Thevenaz (see [CT2]). He also provides strong restrictions on when such representations of dimension greater than one can occur, in terms of the  $p$ -subgroup complex and  $p$ -fusion systems. He recovers and extends in a spectacularly simple fashion several computational results in the literature. The computational potential of his methods is illustrated by calculating  $T_k(G, S)$  and therefore  $T_k(G)$  in other sample new cases, e.g., for the Monster at all primes.

N. Rizo introduced a generalization of Brauer  $p$ -blocks with respect to an invariant character of a normal subgroup, which might give a different perspective and help with the reduction in some of the block theory conjectures, including the  $k(B)$ -conjecture. These blocks, which are related to twisted group algebras, seem to unify several statements, like Brauer’s Height Zero conjecture and the Gluck–Wolf–Navarro–Tiep theorem [NT2].

B. Sambale reported on the efforts to find a counterexample to the defect group Morita equivalent block problem: what is the exact relationship between the defect groups of two Morita equivalent blocks? It has been claimed, and proved in important cases, that these groups might be isomorphic. The experience in group algebra isomorphism problems tell us, though, that there might exist counterexamples but that these might be too big to find, at least with computers.

P. Fong revisited his celebrated paper with Jon Alperin [AF] giving verifications of Alperin’s weight conjecture for symmetric and general linear groups. This paved the way to subsequent verifications of the conjecture for other simple groups but also to the Dade conjecture. The paper has a strong group theoretical side that has been seminal to studies of distinguished classes of nilpotent subgroups of simple groups. But the talk by Fong exemplified rather the program of finding deep connections between combinatorics arising from Alperin’s weight conjecture for the general linear group with the work of Lascoux–Leclerc–Thibon. Indeed a quite classical operation on partitions with a given  $r$ -core was re-interpreted by them in terms of the action of quantum groups on Fock space, thus making a combinatorial bijection more natural. This is of great relevance given how central this combinatorics or their relatives are to representations of finite groups.

A. Paolini lectured on representations of maximal unipotent subgroups  $U$  of finite groups  $G$  of Lie type. He discussed some history and methods about the problem of parametrizing the set  $\text{Irr}(U)$ . He went on to mention some joint results with Goodwin, Le and Magaard about the parametrization of  $\text{Irr}(U)$  when  $G$  is of small rank, including the exceptional types  $F_4$  and  $E_6$ .

H. Nguyen reported on several recent results, obtained by him and his collaborators, that utilize the average degree over a subset of complex irreducible characters of a finite group and generalize various classical theorems in Character Theory of Finite Groups, including theorems of Thompson and Ito–Michler.

The concept of regular embeddings, first formulated by G. Lusztig, is nowadays a standard reduction technique to deal with finite connected reductive groups with disconnected centre. J. Taylor lectured on smooth regular embeddings, a strengthening of this concept which includes the usual embedding of the special linear group into the general linear group. He presented a construction of smooth regular embeddings for any algebraic group, which is done using root data, and then used smooth regular embeddings to provide new proofs of unpublished reduction techniques due to Asai.

Another recent highlight was explained in the talk of Z. Halasi. He reported on the final solution of a 20 year old conjecture by L. Pyber on base sizes for permutation groups. Base sizes for permutation groups have outside applications to computer science. This has been a remarkable tour de force in which many mathematicians have been involved, that required an incredible amount of delicate techniques.

## 4 Scientific Progress Made and Outcome of the Meeting

### 4.1 General Comments

The meeting featured 20 talks, given by well-known experts in the area as well as many younger participants (including postdocs and two Ph. D. students).

Aside from the officially scheduled talks, ample time was allocated to informal discussions. As mentioned above, a discussion session took place on Tuesday afternoon to draw a roadmap towards possible solutions

of some of these principal obstacles on the way to complete proofs of some of the fundamental local-global conjectures in representation theory.

## 4.2 Collaboration Started or Continued During the Meeting

We received an enthusiastic response to our request for feedback from the participants of our meeting which we think illustrates very clearly the many cooperations started or continued during the workshop and how the talks given fostered new ideas and developments.

**R. Boltje:** I was impressed by the high quality of the talks and enjoyed them very much. Benjamin Sambale's talk for instance was very closely related to my own research and gave me some new perspectives. Besides that I would like to mention three more specific developments.

— With Burkhard Külshammer we proved that  $p$ -permutation equivalences between two block algebras  $A$  of a group  $G$  and  $B$  of a group  $H$  preserve a certain family of invariants that is associated to the two blocks. It was already known that a  $p$ -permutation equivalence between  $A$  and  $B$  determines an isomorphism  $\varphi$  between defect groups of  $A$  and  $B$  and that this isomorphism is also an isomorphism between the fusion systems of  $A$  and  $B$ . To each centric  $A$ -Brauer pair  $(P, e)$ , Külshammer and Puig associated an extension of the group  $N_G(P, e)/C_G(P)$  by  $Z(P)$ . This construction is related to the construction of a centric linking system for the fusion system of  $A$ . During the conference, Külshammer and myself proved that if  $(P, e)$  is a centric  $A$ -Brauer pair and  $(Q, f)$  is the corresponding centric  $B$ -Brauer pair (via  $\varphi$ ), then  $\varphi$  also induces an isomorphism between the extension groups associated to  $(P, e)$  and to  $(Q, f)$ .

— With Susanne Danz we started the project to determine the group of perfect self-isometries of a block algebra  $A$  with cyclic defect groups. This is still in the beginning stages.

— Jesper Grodal made some very interesting remarks related to my own talk, saying that the results with my coauthors Kessar and Linckelmann that describe the Picard groups of a block algebra  $B$  use terms that can be derived from homotopy invariants of the centric linking system of the fusion system of  $B$ . We had a very long discussion about these connections that gave me enough incentive to start learning the necessary material from algebraic topology to understand these connections. Although nothing concrete came out of it so far, I'm very excited and hopeful that these connections will give me a much better insight into the topic of my own research.

**C. Bonnafé:** Raphaël Rouquier and me worked on two different subjects. First, we tried to correct an error discovered in our common paper with Jean-Francois Dat on Jordan decomposition of representations of finite reductive groups. This work is going on and will hopefully lead to a positive conclusion. Also, after Rouquier's talk on Hilbert scheme and interesting properties of the canonical line bundle, this suggested to us to look at other blowing-up of varieties of the form  $(V \times V^*)/W$ : we realized that, in some small examples, such blowing-up satisfies analogous properties (one can recover Lusztig's fundamental  $a$ - and  $A$ -invariant geometrically). We intend to work in this direction in the near future.

**S. Danz:** I would like to thank the organisers for organizing this inspiring workshop and for giving me the opportunity to take part in it. As for concrete mathematical discussions and collaborations, I would like to mention the following:

— Thanks to the free afternoon on Wednesday I was able to continue a collaboration with Robert Boltje that we had started this summer. Together with Jürgen Müller we are studying perfect isometry groups of blocks with cyclic defect groups, building on work done by Pornrat Ruengrot (student of Charles Eaton) in his PhD thesis. During the week at BIRS we, in particular, discussed some ideas how to use Boltje's recent work with Markus Linckelmann and Radha Kessar, which he presented in his talk, to make progress on describing these isometry groups.

— After my talk I had some interesting discussions with Benjamin Sambale concerning certain simple  $QG$ -modules that are related to 2-transitive permutation modules, in order to gain new information on integral forms of these modules.

— In his talk on the AWC Paul Fong presented his bijection between the isomorphism classes of simple modules of the symmetric group  $S_n$  in a  $p$ -block  $B$  and the conjugacy classes of  $B$ -weights of  $S_n$ . In doing so, every simple  $B$ -module is, in particular, assigned a certain conjugacy class of  $p$ -subgroups of  $S_n$ . The natural question (also asked by Gabriel Navarro) is whether these groups are related to the vertices of the respective simple modules. It turned out that the group occurring in a  $B$ -weight is not the vertex of the

corresponding simple  $B$ -module, in general. But there might still be a connection, and we are in touch to discuss this.

**C. Eaton:** I think the balance between lectures and free time was about right.

With regards to outcomes: As a direct result of the workshop I'll be visiting Gerhard Hiss to work on the problem of Donovan's conjecture for defect groups  $C_3 \times C_3$ . Whilst in Canada I also discussed this problem with Gunter Malle and Shigeo Koshitani. I discussed the problem of the structure of the Picard group for a block with Robert Boltje, which amongst other things highlighted the importance of parts of his current work with Kessar and Linckelmann. Thanks again for the invitation to the workshop - it was very useful.

**P. Fong:** The conference at BIRS two weeks ago was a very stimulating and enjoyable one — the particular mix of interests and people worked out well. I found the "New Perspectives" in the title of the conference covered not only new mathematical insights, but also the promising young researchers in the representation theory of finite groups who were there.

As for my particular interests: The natural bijections for the Alperin Weight Conjecture in symmetric groups and general linear groups in my talk gave rise to some interesting questions. First, what is the relation of the vertex of an irreducible modular representation of  $S_n$  and the radical subgroup of the weight associated with it? Susanne Danz and Burkhard Külshammer have computed such vertices. We looked at the case  $p = 2$ , but no obvious relation seems to hold. The matter may be different for odd  $p$  and this is being examined now. Another question is whether the natural bijection for symmetric groups passes nicely to alternating groups. This may be a tricky matter - one only has to recall the talk of Olivier Brunat and the inherent difficulties in passing from  $S_n$  to  $A_n$ . But it deserves study. Also of relevance is whether the natural bijection for  $\mathrm{GL}(n, q)$  behaves well with respect to automorphisms. Britta Späth and I had a chance to talk about this — the bijection should behave well and I need to write this up. The question is also relevant to a result of Jiping Zhang and he and I had the opportunity to talk about this. A natural bijection should also exist for unitary groups and some of the ground work has been done by Jianbei An.

Banff also gave me the opportunity to talk with Michel Broué about resuming our collaboration (together with Bhamu Srinivasan) on the Dade conjecture for unipotent blocks in classical reductive groups. The discussion workshop that Britta Späth ran on the inductive local-global conjectures provided an incentive to write-up this project next year during the period our stays at MSRI overlap.

One perspective on the representation theory of finite groups I find particularly intriguing is the application of the representation theory of algebras, be they affine Lie algebras, affine Hecke algebras, Ariki-Koike or any other algebras. That such theories can help settle special problems in finite groups is a fascinating phenomenon. The talk given by Raphaël Rouquier illustrates this well, linking as it did a special case of the decomposition matrix of a unitary group with an algebra over a coefficient domain  $Q(q, t)$  in two transcendental variables. One may hope for more since the representation theory of algebras is still developing.

**E. Giannelli:** The week in Banff has been extremely productive.

In my talk I presented a conjecture concerning the number of linear constituents appearing in the restriction of an irreducible character of a finite group to a Sylow  $p$ -subgroup (this is part of a joint work with Gabriel Navarro). During my stay in Banff I discussed this problem with many researchers that gave me valuable ideas to prove that statement. In particular Michel Broué and Joan Tent suggested different but very promising strategies of proof. This might lead to future collaborations on the topic.

Moreover, I received considerable feedback and suggestions that led me to sensibly improve my analysis of the decomposition into irreducible constituents of the permutation character obtained by inducing the trivial character of a Sylow  $p$ -subgroup of  $S_n$  to the full symmetric group. The paper containing this study is a joint work with Stacey Law and I have been working on it during my stay in Banff. This will be submitted for publication in the next few weeks. The comments of Gerhard Hiss definitely contributed to improving the above mentioned work.

**J. Grodal:** A belated thank you for a great meeting. I really enjoyed it.

It was hugely inspiring to hear the many different approaches to the "standard" conjectures in modular representation theory. Even for the talks which were further away from me, it was interesting to hear what sort of problems people were working with.

In terms of interaction, I spent a good deal of time talking to Raphaël Rouquier about the conjectural role of loop groups in modular representation theory, and with David Craven (and Rouquier) about conjectural

ways to approach Broué's conjecture using subgroup complexes and homotopy theory. We've kept up the dreaming after the conference, but it is still very speculative.

Concretely, I also benefited from Robert Boltje's talk, which seemed to suggest relations to things I've been thinking about earlier. I hope to continue the discussions with him, as well as his collaborators Kessar and Linckelmann at MSRI...

It was also nice to be able to quiz the assembled experts about points in the literature, large and small.

**S. Koshitani:** It is all the time a great pleasure to stay and research Mathematics in BIRS. I have had wonderful experience this time as well.

Actually I was able to discuss the joint work with Caroline Lassueur quite a lot, and therefore our research has had a big improvement. We are researching Puig's Finiteness Conjecture (PFC), that is a more precise conjecture than Donovan's conjecture and that can be stated if we replace "a Morita equivalence" by "a splendid Morita (= Puig) equivalence". The co-worker and me both were suggested by Michel Broué to change (to uniform) the name to a splendid Morita equivalence instead of Puig's equivalence. This was also very helpful.

I could discuss Morita equivalent blocks when our blocks have elementary abelian defect groups of order 9, with Gerhard Hiss and Charles Eaton. This is related to Donovan's Conjecture and is really one of the global-local conjectures in the representation theory of finite groups.

I have learned quite a lot from the many interesting and nice talks held during the week. Finally I really would like to thank the BIRS for giving me such a wonderful opportunity for our research, and also for the four organizers. Hopefully I would like to come back again here in near future.

**C. Lassueur:** The BIRS workshop was very helpful to my current research. In particular a current project with S. Koshitani could be finished and discussions with R. Boltje and C. Eaton brought us some new ideas for future developments of this project. Furthermore, the talk by R. Boltje was particularly relevant to my own research. It let me understand that a result I have with J. Thévenaz on the lifting of endo- $p$ -permutation modules can be extended to modules with fusion-stable endo-permutation sources with a possible application to the lifting of endo-permutation Morita equivalences considered by Boltje–Kessar–Linckelmann through alternative arguments.

**A. Maroti:** Gabriel Navarro and I began to work on a conjecture of his which concerns the possibility of inducing an irreducible character of a nilpotent subgroup of a group to an irreducible character of the group. More precisely, he conjectures that if  $\alpha$  is a complex irreducible character of a nilpotent subgroup of a finite group  $G$  such that  $\alpha^G$  is irreducible, then the generalized Fitting subgroup of  $G$  is nilpotent. We think that we managed to prove a partial result in this direction. Namely, that if  $G$  is a group having the property that it has an irreducible character which can be induced from an irreducible character of a nilpotent subgroup, then the Fitting subgroup of  $G$  is non-trivial or  $G$  is almost simple. There is good evidence that perhaps the second part of the conclusion of this latter claim may be omitted.

At the conference I was happy to see Hung Nguyen for the first time after almost ten years. We had lots of conversations about mathematics. One of these was with Hung and Zoltan Halasi. We had partial results (joint work also with James Cossey) on Gluck's conjecture. This conjecture states that if  $G$  is a finite solvable group,  $b(G)$  is the largest degree of a complex irreducible character of  $G$ , and  $F(G)$  is the Fitting subgroup of  $G$ , then  $|G : F(G)| \leq b(G)^2$ . We had long conversation about how to prove this conjecture. One important special case is when a group has the form  $HV$  where  $V$  is an irreducible and faithful  $H$ -module and  $V$  can only be viewed over a field of size 2, 3, or 4. The difficulty in this case is that there may not exist two vectors  $v$  and  $w$  in  $V$  such that  $C_H(v) \cap C_H(w) = 1$ . This would have been convenient for this latter identity implies that there is a vector  $u$  in  $V$  with  $|C_H(u)|$  at most  $|H|^{1/2}$ , and so there is a large orbit of  $H$  on  $V$ . We tried many ways to get around this difficulty but so far it seems that we will aim to find a (single) vector in  $\text{Irr}(V)$  with a large  $H$ -orbit, at least in as many cases as it is possible, giving us an irreducible character of  $G = HV$  having large degree.

**G. Navarro:** I have now new joint projects with B. Späth and C. Vallejo on the reduction of the McKay-Galois conjecture to finite groups, and several new projects with Pham Huu Tiep, Gunter Malle, Benjamin Sambale and Attila Maroti. Overall, I was impressed by the Rouquier talk. It is quite impressive to witness the announcement of a conjecture of this depth and of this impact. BIRS has become one of the best places to do mathematics and interact with our colleagues in the perfect environment. I want to thank BIRS for making our visit the most enjoyable and fruitful one.

**H. Nguyen:** This was my first time participating in a workshop in BIRS and I had enjoyed every moment of it. I had a chance to discuss some topics and problems that I am particularly interested in with my collaborators. Maroti, Zoltan Halasi and I have a hours-long conversation on a possible completed proof of an old conjecture of Gluck on bounding the index of the Fitting subgroup in terms of the largest character degree. I also discussed with Vallejo about the relation between the average of  $p'$ -degrees of characters in the principal block and the  $p$ -nilpotency of finite groups. During the workshop, Tiep and I were able to finish an ongoing project on an improvement of the Ito–Michler theorem using the notion of average degree. I also had a chance to learn various new perspectives and trends in group representation theory from other participants. I am especially interested in a very recent conjecture proposed by Giannelli and Navarro on the number of distinct linear characters in the restriction of an irreducible character of degree divisible by  $p$  to Sylow  $p$ -subgroups.

**A. Schaeffer Fry:** The BIRS workshop was extremely helpful to my current research. The talks were closely related to my own research and provided me with important updates and new insights to the area.

During the workshop, I made significant progress on a project with J. Taylor, which resulted in finishing and submitting a paper. We also began discussing our next projects.

I also had very fruitful conversations with the organizers and other participants of the workshop. For example, G. Malle suggested a new direction for my research, which led to one of the new projects with Taylor. I spoke with A. Turull about an ongoing project and agreed to keep in touch regarding the problem. I also discussed a potential project with J. Brough, which we have since began work on.

**B. Späth:** The conference was very useful and inspiring. I could meet and discuss the progress of various inductive conditions of the so-called global-local conjectures. It was very interesting to meet Paul Fong and Jiping Zhang. With Paul Fong I discussed his recent refinements of his deep but older results on the Alperin weight conjecture for symmetric and finite general linear groups. Also Jiping Zhang presented several results of his in this direction. The discussions clarified the current difficulties with the inductive conditions. Also it was useful to get informed on recent results of Jianbei An and Gerhard Hiss. Meeting these experts and discussing their on-going research project gave me the feeling that a new approach to the inductive conditions will be needed. I started thinking about that I further discussed in this context with Cedric Bonnafé the action on unipotent characters that might simplify the proof of a recent result joint with Marc Cabanes.

Several talks showed the interest in the Galois refinement of the McKay conjecture. This encouraged me to start a project with Gabriel Navarro and Carolina Vallejo in this direction.

The workshop was made very comfortable thanks to the BIRS center providing a very convenient family suite and babysitter time for my family, Marc and our baby son Maurice. The breakthrough I may remember most is the fact that Maurice started to walk for good that very week in the corridors of Corbett Hall!

**J. Taylor:** This present meeting at Banff was one of the most useful meetings I have attended in my career thus far. After my talk on root data I had a lengthy and interesting discussion with Frank Lübeck about how one could implement these theoretical ideas computationally. I hope this will result in something useful. My collaborator, Mandi Schaeffer Fry, and I managed to complete our current project on principal 2-blocks, which appeared on the arXiv shortly after the meeting. In discussions with Gunter Malle and Pham Tiep we came up with some new ideas on our current joint project, which will hopefully be successful. Alex Turull shared with me his work on the Glauberman correspondence and pointed me towards an open problem on Green Functions for finite reductive groups, which I am still thinking about. Finally, I was really encouraged by the number of younger participants at the meeting. I think this is a great testament to the depth and fortitude of finite groups as an area of study.

**P. Tiep:** I have had fruitful discussions with G. Malle and J. Taylor on our ongoing joint project. We have also started new joint projects with G. Navarro and with H. N. Nguyen.

**A. Turull:** I was very impressed by the organization, and how well the facilities were administrated. The scientific aspects of the conferences were likewise excellent. The topics of the conference had a focus on new perspectives in the representation theory of finite groups. It was interesting for me to see and hear what other experts on this are thinking currently. There is currently progress on the many important conjectures in representation theory, and it was very useful for me to hear the details of their perspectives. I was particularly interested to hear some work on the Navarro Conjecture for particular groups. I think this may lead to the proof of the stronger version of this conjecture that I proposed for those same groups. I was also interested to discuss some new research avenues with some of the younger participants. It was an excellent and productive experience all around.

**C. Vallejo:** Thanks for making possible such a great week at BIRS. It has been an especially enjoyable workshop for me. Despite not having given a talk, I have found it very fruitful. I have talked with Nguyen about the possibility of extending some of the results he presented in his talk by considering the average of degrees of characters in the principal block and not in the whole group. Also, I have had the opportunity to talk with Britta Späth about Navarro's conjecture. She has explained to me a new way, that she recently discovered, of relating central isomorphisms of character triples with Galois action. This might lead to a reduction theorem for Navarro's conjecture in a near future.

**J. Zhang:** Banff workshop was just excellent.

I talked with many participants and discussed with Fong, Broué, Koshitani, etc. on parametrizations of characters of groups of Lie type related to the Alperin–McKay conjecture. In my Banff talk I mentioned our verification of the Inductive Blockwise Alperin weight condition for  $\mathrm{PSL}(3, q)$ . However there exists a small inaccuracy in a result on radical  $p$ -subgroups we cited, however this is ok for  $G = \mathrm{PSL}(3, q)$ . The paper based on my Banff talk is now accepted by J. Alg. Encouraged by the Banff workshop we will continue to verify further Inductive Conditions. During the Banff workshop I also discussed with Külshammer, Nguyen Hung, etc. about further research collaborations. Külshammer will visit us for a month next year, and Nguyen Hung and I will possibly organize a research program on character degree problems in 2018–2019.

All the workshop's participants agreed that Banff lived up to its promises of a quiet, inspiring and very comfortable place to make science. We are all very grateful to the BIRS for providing such excellent facility for discussing and doing mathematics, and hope to return some time in the future.

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