A substitute for square lattice designs with 36 treatments

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Joint work with Peter Cameron (University of St Andrews) and Tomas Nilson (Mid-Sweden University)

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I will describe the design, and say something about its properties.

Outline

1. Square lattice designs.
2. Triple arrays and sesqui-arrays.
3. How the new designs were discovered, part I.
4. Resolvable designs for 36 treatments in blocks of size 6.
5. How the new designs were discovered, part II.

Chapter 1

Square lattice designs.

Square lattice designs for 16 treatments in 2–4 replicates

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

A	B	C	D
B	A	D	C
C	D	A	B
D	C	B	A

α	β	γ	δ
γ	δ	α	β
δ	γ	β	α
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Replicate 1

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

Replicate 2

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All pairwise treatment concurrences are in $\{0,1\}$.

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2. The blocks of Replicate 1 are given by the rows;
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3. If $r = 2$ then STOP.
4. Otherwise, write down $r - 2$ mutually orthogonal Latin squares of order n.
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Cheng and Bailey (1991) showed that these designs are optimal among block designs of this size, even over non-resolvable designs.

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Patterson and Williams (1976) used computer search to find a design for 36 treatments in 4 replicates of blocks of size 6. All pairwise treatment concurrences are in $\{0, 1, 2\}$.
The value of its A-criterion is 0.836, which compares well with the unachievable upper bound of 0.840.

Chapter 2

Triple arrays and sesqui-arrays.

Triple arrays

Triple arrays were introduced independently by Preece (1966) and Agrawal (1966), and later named by McSorley, Phillips, Wallis and Yucas (2005).

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A triple array with $r = 4$, $c = 9$, $v = 12$ and $k = 3$

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Sterling and Wormald (1976) gave this triple array.

D	H	F	L	E	K	I	G	J
A	K	I	B	J	G	C	L	H
J	A	L	D	B	F	K	E	C
G	E	A	H	I	B	D	C	F

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If letters are blocks, rows are levels of treatment factor $T1$, columns are levels of treatment factor $T2$, and there is no interaction between $T1$ and $T2$, then this is a good design.

Sesqui-arrays are a weakening of triple arrays

Cameron and Nilson introduced the weaker concept of sesqui-array by dropping the condition on pairs of columns. They are row–column designs with r rows, c columns and v letters, satisfying the following conditions.

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Chapter 3

How the new designs were discovered, part I.

The story: Part I

Consider designs with $n+1$ rows, n^2 columns and $n(n+1)$ letters. Triple arrays have been constructed for $n \in \{3,4,5\}$ by Agrawal (1966) and Sterling and Wormald (1976); for $n \in \{7,8,11,13\}$ by McSorley, Phillips, Wallis and Yucas (2005). There are values of n, such as $n=6$, for which a BIBD for n^2 treatments in $n(n+1)$ blocks of size n does not exist.

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Later, RAB found a simpler version of TN's construction, that needs a Latin square of order n but not orthogonal Latin squares. So $n=6$ is covered. If this had been known earlier, PJC would not have found the nice design for $n=6$.

Chapter 4

Resolvable designs for 36 treatments in blocks of size 6.

The Sylvester graph and its spiders

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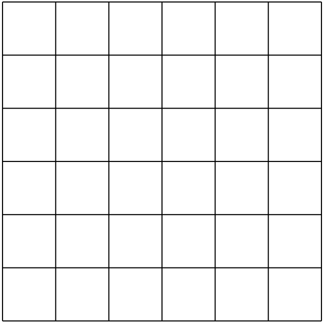
The vertices can be thought of as the cells of a 6×6 grid.



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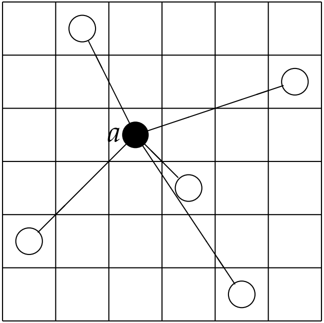

At each vertex a, the *spider* $S(a)$ defined by the 5 edges at a has 6 vertices, one in each row and one in each column.

Spiders whose centres are in the same column

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					c
		a			

If there is an edge from a to c and an edge from b to c then the spider $S(c)$ has two vertices in the third column.

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So, for any one column,
the 6 spiders centred on vertices in that column do not overlap,
and so they give a single replicate of 6 blocks of size 6.

Constructing resolved designs with r replicates

For $r = 2$ or $r = 3$:

Replicate 1	the blocks are the rows of the grid
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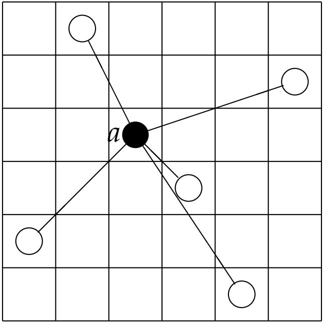
The fine details of which designs we chose do not fit in the margin.

More properties of the Sylvester graph



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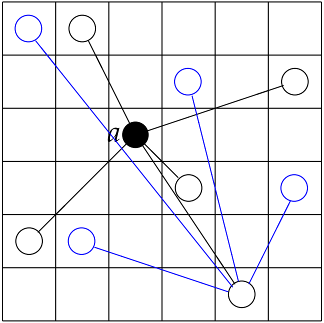

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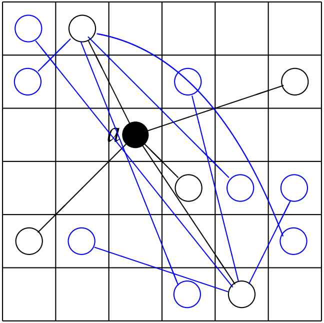

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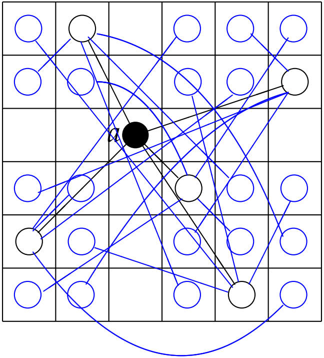


Vertices at distance 2 from a are all in rows and columns different from a.
The Sylvester graph has no triangles or quadrilaterals.

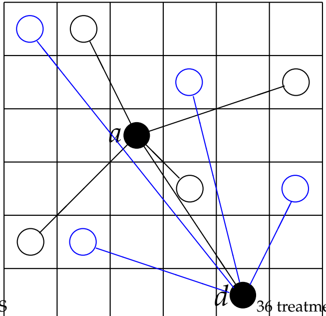
This implies that, if a is any vertex, the vertices at distance 2 from vertex a are precisely those vertices which are not in the spider $S(a)$ or the row containing a or the column containing a.

Consquence I: concurrences

If a is any vertex, the vertices at distance 2 from vertex a are precisely those vertices which are not in the spider $S(a)$ or the row containing a or the column containing a.

Consequence

If we make each spider into a block, then the only way that distinct treatments a and d can occur together in more than one block is for vertices a and d to be joined by an edge so that they both occur in the spiders $S(a)$ and $S(d)$.



Consquence II: association scheme

If a is any vertex, the vertices at distance 2 from vertex a are precisely those vertices which are not in the spider $S(a)$ or the row containing a or the column containing a.

Consequence

The four binary relations:

- different vertices in the same row;
- different vertices in the same column;
- vertices joined by an edge in the Sylvester graph Σ;
- vertices at distance 2 in Σ

form an association scheme.

Consequence II: association scheme

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form an association scheme.

So, for any incomplete-block design which is partially balanced with respect to this association scheme, the information matrix has five eigenspaces, which we know (in fact, they have dimensions 1, 5, 5, 9 and 16), so it is straightforward to calculate the eigenvalues and hence the canonical efficiency factors.

Constructing a resolved design with 6 replicates

For each column, make a replicate whose blocks are the 6 spiders whose centres are in that column.

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$$\text{concurrence} = \begin{cases} 2 & \text{for vertices joined by an edge} \\ 1 & \text{for vertices at distance 2} \\ 0 & \text{for vertices in the same row or column.} \end{cases}$$

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The unachievable upper bound given by the non-existent square lattice design is $A = 0.8537$.

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The harmonic mean is $A = 0.8507$.

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canonical efficiency factor	1	$\frac{19}{21}$	$\frac{6}{7}$	$\frac{11}{14}$
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The harmonic mean is $A = 0.8507$.

The unachievable upper bound given by the non-existent square lattice design is $A = 0.8571$.

Constructing a resolved design with 8 replicates

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canonical efficiency factor	$\frac{11}{12}$	$\frac{7}{8}$	$\frac{13}{16}$
multiplicity	9	10	16

The harmonic mean is $A = 0.8676$.

The non-existent design consisting of a balanced design in 7 replicates with one more replicate adjoined would have $A = 0.8547$.

Chapter 5

How the new designs were discovered, part II.

Back to the sesqui-arrays

These wonderful designs are a fortunate byproduct of a wrong turning in the search for sesqui-arrays.

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How do we take the one with 7 replicates and turn its dual into a 7×36 sesqui-array with 42 letters?

The story: Part II

RAB: I am typing up some of these new designs. Is your sesqui-array for $n = 6$ written out explicitly?

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Each column has the correct set of letters,
but their arrangement in rows is wrong,
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Later, PJC: The only hope of putting this right is to permute the letters in each column. I need 6 permutations. Each fixes the first row and one other. The rest of each permutation gives a circle on the other 5 rows, and I want these circles to have every row following each other row exactly once.

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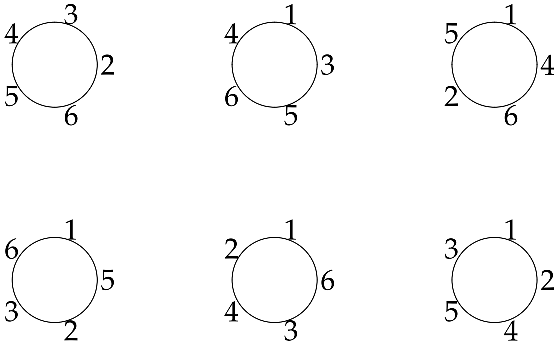
RAB: Easy peasy. That is a neighbour-balanced design for 6 treatments in 6 circular blocks of size 5. I made one of those for experiments in forestry 25 years ago.

And so the sesqui-array for $n = 6$ was constructed.

That forestry design that we used



Ongoing work

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This is harder than the above, because we cannot use the association scheme if we are not using all spiders.
On the other hand, the calculation is made easier by the fact that, because of the large group of automorphisms,
if we use the spiders from m columns (where $1 \leq m \leq 5$)
it does not matter which subset of m columns we use.

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