A toolbox for clustering properties of Macdonald polynomials

Laura Colmenarejo

York University - Fields Institute
with J-G. Luque and C. F. Dunkl.

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Theoretical Physics

Many-body problem
Quantum Hall Effect

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Many-body problem
Quantum Hall Effect

$\Longrightarrow$

Combinatorics

Expand the powers of the discriminant on Schur functions

Discriminant

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- Square of the Vandermonde determinant

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- Classification of entanglement: use the (geometric) invariant theory to classify quantum systems of particles (qubit systems)

Back to Physics: Jack and Macdonald polynomials

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- Form of the eigenfunctions and eigenvectors involved
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- Generalization of the Jack polynomials: $q = t^{\alpha}$ and $t \longrightarrow 1$
- Not only one, but four versions (for each!)

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Non-symmetric	Shifted	Macdonald poly.
Symmetric	Homogeneous	Jack poly.

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We compute the **non–symmetric shifted Macdonald polynomial** associated to the vector $[2, 1, 0]$ and we get this nice result

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&-\frac{1}{(qt-1)^3(qt+1)q}\big(q^5t^8 - q^5t^7x1 - q^5t^7x2 + q^5t^6x1x2 - q^4t^7x3 + q^4t^6x1x3 + q^4t^6x2x3 - q^4t^6x1 + q^4t^5x1^2 - q^4t^5x1x3 - q^4t^5x2x3 - q^4t^4x1^2x2 - q^4t^6 \qquad (5)\\
&+ 2q^4t^5x1 + 2q^4t^5x2 + q^4t^5x3 - q^4t^4x1x2 - q^3t^6x2 - q^3t^6x3 + q^3t^5x1x2 + 2q^3t^5x1x3 + q^3t^5x2^2 + q^3t^5x2x3 - q^3t^4x1^2x3 - q^3t^4x1x2^2 - q^3t^4x1x2x3 - q^4t^5\\
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&- q^2t^4x3^2 + q^2t^3x1^2x3 + q^2t^3x1x2^2 + 5q^2t^3x1x2x3 + 2q^2t^3x1x3^2 + 2q^2t^3x2^2x3 + q^2t^3x2x3^2 - q^3t^3x1 - q^3t^3x2 - q^3t^3x3 + q^2t^4x2 + 2q^2t^4x3 - q^2t^3x1^2\\
&- 3q^2t^3x1x2 - 3q^2t^3x1x3 - q^2t^3x2^2 - q^2t^3x2x3 - q^2t^2x1^2x3 - q^2t^2x1x2^2 - 3q^2t^2x1x2x3 - q^2t^2x1x3^2 - q^2t^2x2^2x3 + qt^4x2x3 + 2qt^3x2x3^2 + q^3t^3 - q^2t^3x2\\
&- q^2t^3x3 + q^2t^2x1^2 + 4q^2t^2x1x2 + 3q^2t^2x1x3 + q^2t^2x2^2 + 2q^2t^2x2x3 - qt^3x1x2 - 2qt^3x1x3 - qt^3x2^2 - 4qt^3x2x3 - 2qt^3x3^2 + qt^2x1^2x3 + qt^2x1x2^2\\
&+ 2qt^2x1x2x3 - 2qt^2x2x3^2 - t^3x2x3^2 + q^2t^3 - 2q^2t^2x1 - 2q^2t^2x2 - q^2t^2x3 + qt^3x2 + qt^3x3 + qt^2x1^2 + qt^2x1x2 + 2qt^2x1x3 + qt^2x2^2 + 4qt^2x2x3\\
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&+ 2qtx1x2 + qtx1x3 + qx1x2x3 - t^2x2x3 - tx1^2x3 - tx1x2^2 - 3tx1x2x3 - 2tx1x3^2 - 2tx2^2x3 - 2tx2x3^2 - qtx1 - qx1x2 - qx1x3 - qx2x3 + tx1x2\\
&+ 2tx1x3 + tx2^2 + 3tx2x3 + tx3^2 + x1^2x2 + x1^2x3 + x1x2^2 + 2x1x2x3 + x1x3^2 + x2^2x3 + x2x3^2 + qx1 + qx2 + qx3 - tx2 - tx3 - x1^2 - 2x1x2 - 2x1x3\\
&- x2^2 - 2x2x3 - x3^2 - q + x1 + x2 + x3\big)
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\end{aligned}
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But if we consider the specialization given by $qt^2 = 1$, then

$$M_{[2,1,0]}\Big|_{q=\frac{1}{t^2}} = -t^2(tx_3 - x_2)(tx_3 - x_1)(tx_2 - x_1)$$

Affine Hecke algebra of the symmetric group

$$\mathcal{H}_N(q,t) = \mathbb{C}(q,t)\left[x_1^{\pm},\ldots,x_N^{\pm},T_1^{\pm},\ldots,T_{N-1}^{\pm},\tau\right]$$

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The operators T_i satisfy the relations of the Hecke algebra in the symmetric group:

- $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ (braid relation)
- $T_i T_j = T_j T_i$, for $|i-j| > 1$
- $(T_i - t)(T_i + 1) = 0$

Operators and polynomials

(q, t)–Cherednik operators

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Non–symmetric shifted Macdonald polynomials $(M_v)_{v \in \mathbb{N}^N}$: unique basis of simultaneous eigenfunctions of the operators Ξ_i.

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- Eigenvalues: spectral vectors $Spec(v)$
- Yang-Baxter graph: provides a method to compute non-symmetric (shifted) Macdonald polynomials

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- Affine operation: $M_{v.\Phi} = M_v\tau(x_N - 1)$
- Vanishing properties:
  - $M_v\left(\langle u\rangle\right) = 0$ for $|v| \leq |u|$, $u \neq v$
  - $M_v\left(\langle v\rangle\right) = \pm t^{\star} h_{t,q}(v,q)$, where $h_{q,t}(v,z)$ is the $(q,t)-$hook product of v

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We can find many examples of singular Macdonald polynomials, as well as conjectures, but the general form remains unknown.

Solution for the staircase

Theorem (Dunkl, Luque, C. - 2015)

Let $v_{n,k} = [(n-1)k, (n-2)k, \ldots, k, 0]$. *Consider the specialization* $q^k t^2 = 1$, *with* k *odd or* $q^{\frac{k}{2}} t \neq 1$. *Then,*

$$M_v(q,t) = E_v(q,t) = \pm t^{\star} \prod_{l=1}^{k} \prod_{i<j} \left(x_i - \frac{1}{tq^l} x_j \right)$$

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Examples

$$M_{[2,1,0]}\big|_{q=\frac{1}{t^2}} = -t^2 (tx_3 - x_2)(tx_3 - x_1)(tx_2 - x_1)$$

$$M_{[4,2,0]}\big|_{q=\frac{-1}{t}} = t^7 (x_1 - tx_2)(x_1 - tx_3)(x_2 - tx_3)(x_1 + x_2)(x_1 + x_3)(x_2 + x_3)$$

Solution for the quasi-staircase?

Conjecture (Dunkl, Luque, C.)

Let $v = v_{n,k,\alpha,\beta}$ *be the quasi–staircase. Consider the specialization* $q^k t^{\alpha+1} = 1$, *with* $g = 1$ *or* $q^{\frac{k}{g}} t^{\frac{\alpha+1}{g}} \neq 1$, *where* $g = \mathrm{gcd}(k, \alpha+1)$. *Then,*

$$\begin{aligned} M_v\left(x_1, \dots, x_\beta, y_1, \dots, t^{\alpha-1}y_1, \dots, t^{\alpha-1}y_{n-1}\right) = \\ = E_v\left(x_1, \dots, x_\beta, y_1, \dots, t^{\alpha-1}y_1, \dots, t^{\alpha-1}y_{n-1}\right) = \\ = \pm t^{\star} \prod_{l=1}^{k} \left[\left[\prod_{i=1}^{\beta} \prod_{j=1}^{n-1} (x_i - \frac{1}{tq^l} y_j) \right] \left[\prod_{s=1}^{\alpha} \prod_{i<j} (t^s y_i - \frac{1}{tq^l} y_j) \right] \right] \end{aligned}$$

Examples

$$M_{21100}\left(x_1, y_1, ty_1, y_2, ty_2; \frac{1}{t^3}, t\right) =$$
$$= t^4(y_1 - t^2y_2)(y_1 - ty_2)(x_1 - t^2y_2)(x_1 - t^2y_1).$$

$$M_{42200}\left(x_1, y_1, z^2y_1, y_2, z^2y_2; \frac{1}{z^3}, z^2\right) =$$
$$= z^{27}(y_1 - zy_2)(y_1 - z^2y_2)(y_1 - z^4y_2)(zy_1 - y_2)$$
$$(x_1 - zy_2)(x_1 - z^4y_1)(x_1 - zy_1)(x_1 - z^4y_1).$$

$$M_{300}\left(x_1, y_1, zy_1; \frac{\omega}{z}, z\right) =$$
$$= \frac{-1}{4}z^3(x_1 - z^2y_1)(2x_1 + y_1 + i\sqrt{3}y_1)(-2x_1 - zy_1 + i\sqrt{3}zy_1),$$

where $\omega = \frac{-1}{2} + \frac{1}{2}\sqrt{3}i$.

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  - If λ has no zero parts, we can describe M_λ in terms of $M_{\lambda-(\lambda_N,\dots,\lambda_N)}$ (up to a factor)

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- The difficult part is to prove the singularity of M_v.
- Other results:
  - If λ has no zero parts, we can describe M_λ in terms of $M_{\lambda-(\lambda_N,\dots,\lambda_N)}$ (up to a factor)
  - If λ has zero parts, we can consider a *standard specialization*. For instance,

$$M_{32000}(x_1, x_2, t^2, t, 1) \stackrel{(*)}{=} M_{32}\left(\frac{x_1}{t^3}, \frac{x_2}{t^3}\right)$$

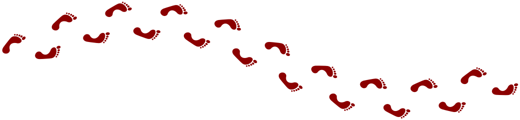
Final remarks

- The difficult part is to prove the singularity of M_v.
- Other results:
  - If λ has no zero parts, we can describe M_λ in terms of $M_{\lambda-(\lambda_N,\ldots,\lambda_N)}$ (up to a factor)
  - If λ has zero parts, we can consider a *standard specialization*. For instance,

$$M_{32000}(x_1, x_2, t^2, t, 1) \stackrel{(*)}{=} M_{32}\left(\frac{x_1}{t^3}, \frac{x_2}{t^3}\right)$$

- Can we describe all the *nice* specializations?

Thank you very much!



¡Muchas gracias!

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