The Peterson Isomorphism: Moduli of Curves and Alcove Walks

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Folded alcove walks are useful in many familiar contexts:

- Schubert calculus of all flavors (classical, quantum, affine, equivariant, K-theory)
- Calculating weight multiplicities using crystals
- Determining which Kostka numbers are nonzero (expanding Schurs in terms of the monomial basis)
- Newton saturation property of many polynomials arising in algebraic combinatorics
- Doing calculations in any Hecke algebra (finite, affine, double)
- Rook placements and Jordan forms (?)

By no means an exhaustive list, just one which connects to ideas already discussed at this particular workshop

Flag Varieties

Notation:

- G split connected reductive group over $\mathbb{C}$
- Fix a Borel containing a split maximal torus $G \supset B \supset T$
- The opposite Borel subgroup is B^-
- W is the finite Weyl group $N_G(T)/T$

Example ($G = \mathrm{SL}_3$)

B is upper-triangular matrices and T is the diagonal matrices:

$$B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \supset T = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}$$

$$B^- = \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\} \qquad s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in W = S_3.$$

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Definition

The *flag variety* is the quotient G/B.

Fact (Bruhat Decomposition)

The flag variety G/B decomposes into *Schubert cells*

$$G(\mathbb{C}) = \bigsqcup_{u \in W} BuB = \bigsqcup_{v \in W} B^- vB.$$

Flag Varieties

The Affine Context:

- $R = \mathbb{C}[t, t^{-1}] \supset \mathcal{O} = \mathbb{C}[t] \supset \mathcal{O}^\times = \mathbb{C}^\times$
- Iwahori subgroup I of $G(R)$ is the preimage of $B(\mathbb{C})$ under the projection $G(\mathcal{O}) \to G(\mathbb{C})$ by $t \mapsto 0$

Example ($G = \mathrm{SL}_3$)

If B is upper-triangular matrices, the Iwahori subgroup equals

$$I = \left\{ \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} & \mathcal{O} \\ t\mathcal{O} & \mathcal{O}^\times & \mathcal{O} \\ t\mathcal{O} & t\mathcal{O} & \mathcal{O}^\times \end{pmatrix} \right\} \subset G(R)$$

Definition

The *affine flag variety* is the quotient $G(R)/I$.

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We can also study the affine flag variety by carving it up using an appropriate analog of the Bruhat decomposition.

Fact (Affine Bruhat Decomposition)

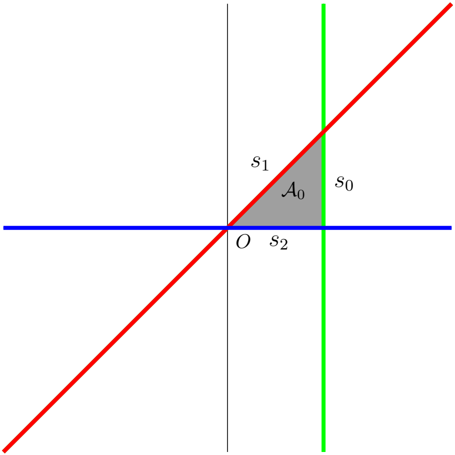
The affine flag variety decomposes into *affine Schubert cells*

$$G(R) = \bigsqcup_{x \in \widetilde{W}} IxI,$$

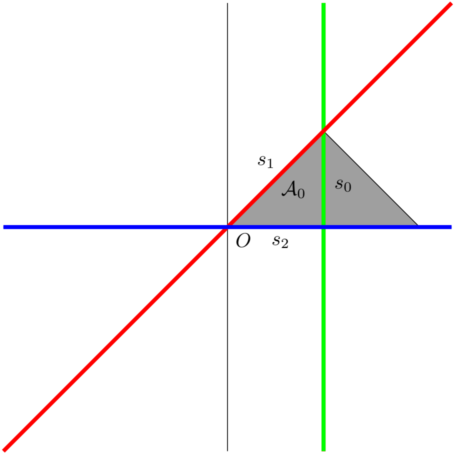
where $\widetilde{W}$ is the *affine Weyl group*.

Affine Weyl Group

Example: $G = Sp_4$

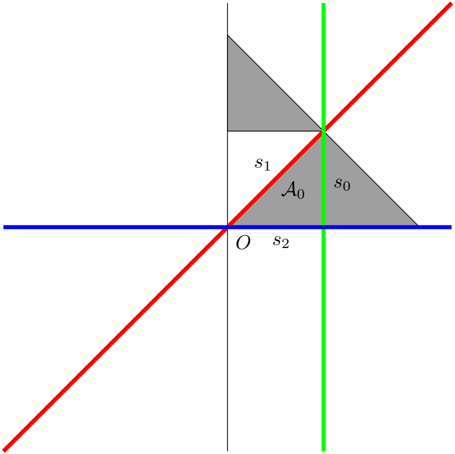


The additional generator s_0 is an affine reflection.



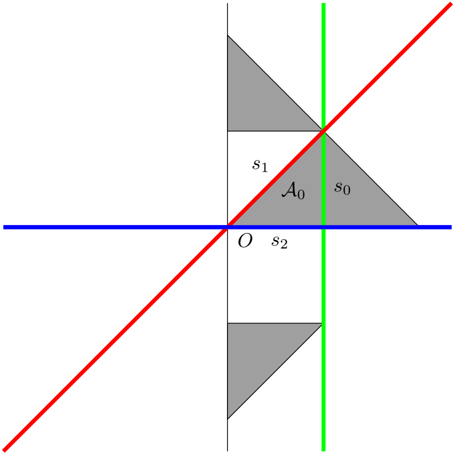
The result of applying s_0 to the base alcove $\mathcal{A}_\circ \longleftrightarrow 1$.

Affine Weyl Group



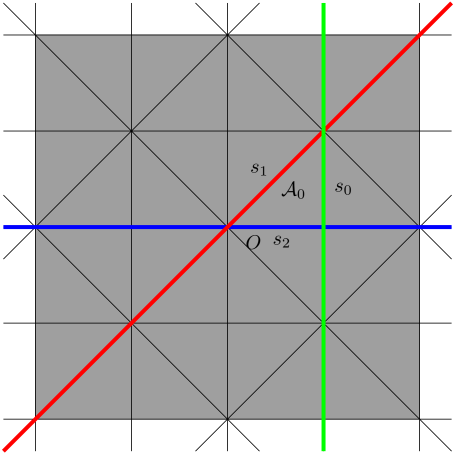
The result of applying s_1 to $s_0(\mathcal{A}_\circ)$ is $s_1 s_0(\mathcal{A}_\circ)$.

Affine Weyl Group



The result of applying s_2 to $s_1 s_0(\mathcal{A}_\circ)$ is $s_2 s_1 s_0(\mathcal{A}_\circ)$.

Affine Weyl Group

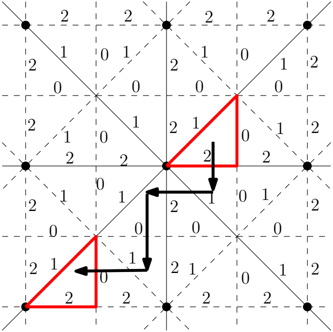


Elements of the affine Weyl group $\widetilde{W}$ correspond to alcoves.

Labeled Folded Alcove Walks

Definition

An *alcove walk* is a sequence of moves from an alcove to an adjacent alcove obtained by crossing an affine hyperplane.



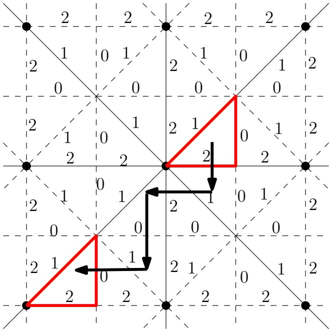
An alcove walk corresponding to the word $s_2 s_1 s_2 s_0 s_1 s_0$.

$$\{\text{alcove walks}\} \longleftrightarrow \left\{\text{words in } \widetilde{W}\right\}$$

Labeled Folded Alcove Walks

Theorem (Steinberg 1967, Parkinson-Ram-Schwer 2009)

$$\{\textit{labeled alcove walks}\} \longleftrightarrow \{\textit{double cosets } IxI\}$$

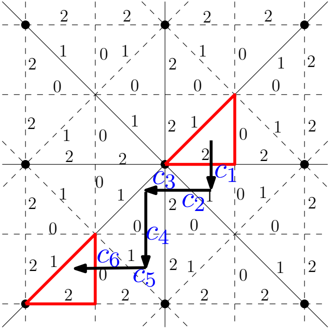


A minimal length alcove walk to s_{212010}.

Labeled Folded Alcove Walks

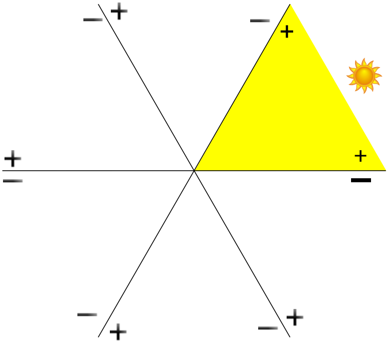
Theorem (Steinberg 1967, Parkinson-Ram-Schwer 2009)

$$\{\textit{labeled alcove walks}\} \longleftrightarrow \{\textit{double cosets } IxI\}$$



All points of Sp_4/I in the affine Schubert cell $Is_{212010}I$.

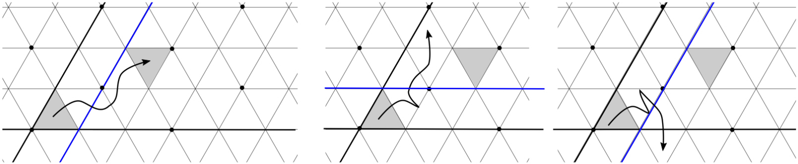
For each $x \in \widetilde{W}$, the orientation induced by x is defined so that alcove x is on the positive side of every affine hyperplane.



Labeled Folded Alcove Walks

Definition

Orient the hyperplanes so that the identity alcove $\mathcal{A}_\circ$ is on the positive side of every affine hyperplane. A fold is a *positive fold* if it occurs on the positive side of a hyperplane.



Rules for creating folded alcove walks:

1. Can only do positive folds.
2. Must fold from tail-to-tip.

Labeled Folded Alcove Walks

Definition

For $x, y \in \widetilde{W}$, define $\mathcal{A}_\circ(x, y)$ to be the set of all labeled folded alcove walks γ such that:

- γ is positively folded with respect to the orientation induced by $\mathcal{A}_\circ$,
- γ is obtained by folding a minimal walk from 1 to x, and
- γ ends in the y alcove.

Labeled folded alcove walks see intersections of Schubert cells.

Theorem (adaptation of Parkinson-Ram-Schwer 2009)

$$\mathcal{A}_\circ(x, y) \stackrel{\sim}{\longleftrightarrow} IxI \cap I^- yI$$

Application: Moduli of Curves

The moduli space of genus 0 curves on G/B decomposes as

$$\mathcal{M}_3 = \bigsqcup_{\tau \in Q^\vee} \bigsqcup_{u,v \in W} \mathcal{M}_{3,\tau}^{u,v}, \quad \text{where}$$

$$\mathcal{M}_{3,\tau}^{u,v} = \left\{ C : \mathbb{P}^1 \to G/B \;\middle|\; \begin{array}{l} C_*([\mathbb{P}^1]) = \tau, \\ C(0) \in BuB, \\ C(\infty) \in B^- vB \end{array} \right\}.$$

Theorem (Peterson, M.–Ram)

$$\mathcal{A}_\circ(ut_{\infty\rho^\vee}, vt_{\infty\rho^\vee+\tau}) \xleftrightarrow{\sim} \mathcal{M}_{3,\tau}^{u,v}$$

Remarks: Each labeled folded alcove walk ...

- gives an explicit algorithm for writing down a rational parameterization for the corresponding curve in G/B,
- contributes a term in "Billey's Formula" for $H_T^*(G/I)$.

Thank you!