A Remmel-Whitney style rule for quasisymmetric Schur functions

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Schur and quasisymmetric Schur functions

Schur functions

1. are symmetric
2. are a basis for the ring of symmetric functions
3. have a nice combinatorial definition
4. appear in many other areas, including representation theory, algebraic geometry

Quasisymmetric Schur functions

1. are quasisymmetric
2. are a basis for the ring of quasisymmetric functions
3. have a nice combinatorial definition
4. refine the (symmetric) Schur functions

Semi-standard Young tableaux

Let λ be an integer partition. A semi-standard Young tableau of shape λ is a filling of the diagram of λ with positive integers so that columns strictly increase from bottom to top and rows weakly increase from left to right.

$T =$

5	5			
3	4	4		
1	2	3	3	4

The *content monomial* of T is $x^T = \prod_i x_i^{\text{\# of } i\text{'s in } T}$.

$$x^T = x_1 x_2 x_3^3 x_4^3 x_5^2$$

Schur functions

Given an integer partition λ, the Schur function indexed by λ is

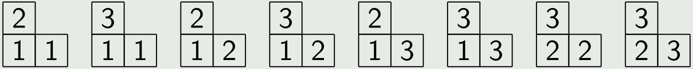
$$s_\lambda = \sum_T x^T$$

where the sum is over all semi-standard Young tableaux of shape λ.

Example when $\lambda = (2,1)$

$$s_{(2,1)}(x_1, x_2, x_3) =$$

$$x_1^2x_2 \; + \; x_1^2x_3 \; + \; x_1x_2^2 \; + \; 2x_1x_2x_3 \; + \; x_1x_3^2 \; + \; x_2^2x_3 \; + \; x_2x_3^2$$



Littlewood-Richardson Rule

Theorem

For partitions λ and μ,

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$$

where the sum is over all partitions ν such that $|\nu| = |\lambda| + |\mu|$ and $\mu \subseteq \nu$. The coefficients, $c^\nu_{\lambda\mu}$, are the number of Littlewood-Richardson tableaux of shape ν/μ and content λ.

Theorem (Remmel, Whitney 1984)

There exists a set $\mathcal{O}(\lambda, \mu, \nu)$ of standard tableaux such that

- $|\mathcal{O}(\lambda, \mu, \nu)| = c^\nu_{\lambda\mu}$, *and*
- *the elements of $\mathcal{O}(\lambda, \mu, \nu)$ can be generated algorithmically as leaves of a certain tree.*

Quasisymmetric Polynomials

Quasisymmetric polynomials

A polynomial p is *quasisymmetric* if

$$\text{coeff.}\left(x_1^{a_1}\, x_2^{a_2} \cdots x_k^{a_k}\right) \;=\; \text{coeff.}\left(x_{i_1}^{a_1}\, x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}\right)$$

for all $i_1 < i_2 < \cdots < i_k$.

Example.

$$x_1^2 x_2^4 + x_1^2 x_3^4 + x_2^2 x_3^4 \quad \text{and} \quad x_1^4 x_2^2 + x_1^4 x_3^2 + x_2^4 x_3^2$$

Quasisymmetric Schur Functions

Composition tableaux

Given a composition α, a composition tableau is a filling, F, of the cells of the diagram of α such that

1. The leftmost column entries strictly increase from bottom to top.
2. The row entries weakly increase from L to R.
3. The entries satisfy the **triple rule:**

if $a \geq b$, then $a > c$

b	c
	a

$F =$

3	3	4
2	2	
1	1	5

$x^F = x_1^2 x_2^2 x_3^2 x_4 x_5$

Quasisymmetric Schur Functions

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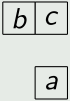


The quasisymmetric Schur function indexed by α is

$$\mathcal{C}_\alpha = \sum_F x^F$$

where the sum is over all composition tableaux of shape α.

- The $\mathcal{C}_\alpha$ are quasisymmetric.
- They refine the Schur functions:

$$s_\lambda = \sum_{\widetilde{\alpha}=\lambda} \mathcal{C}_\alpha$$

- They form a basis for the quasisymmetric functions.
- Behave similarly to Schur functions.

Littlewood-Richardson Rule

Theorem (Haglund et al.)

Let μ be a partition and α a composition. Then

$$\mathcal{C}_\alpha s_\mu = \sum_\beta A^\beta_{\alpha,\mu} \mathcal{C}_\beta$$

where $|\beta/\alpha| = \mu$ and $A^\beta_{\alpha,\mu}$ is the number of Littlewood-Richardson composition tableaux of shape β/α and content μ.

Definition

A Littlewood Richardson composition tableau is a skew composition tableau of shape β/α with the properties:

1. rows weakly increase from left to right,
2. the column reading word (down columns starting with rightmost) is a lattice word, and
3. two triple conditions are satisfied.

Remmel-Whitney Rule

Definition

Let α be a composition and λ be a partition. Then

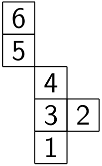
$$\alpha * \lambda := (\lambda_1 + \alpha_1, \lambda_1 + \alpha_2, \ldots, \lambda_1 + \alpha_k, \lambda_1, \lambda_2, \ldots, \lambda_m)/(\lambda_1)^k$$

and define $S_{\alpha*\lambda}$ to be the filling of $\alpha * \lambda$ obtained by placing the labels $1, 2, \ldots, |\alpha| + |\lambda|$ into the diagram of $\alpha * \lambda$ in reverse reading order.

Example.

Let $\alpha = (1, 2, 1)$ and $\lambda = (1, 1)$. Then

$S_{\alpha*\lambda} =$



Remmel-Whitney Rule

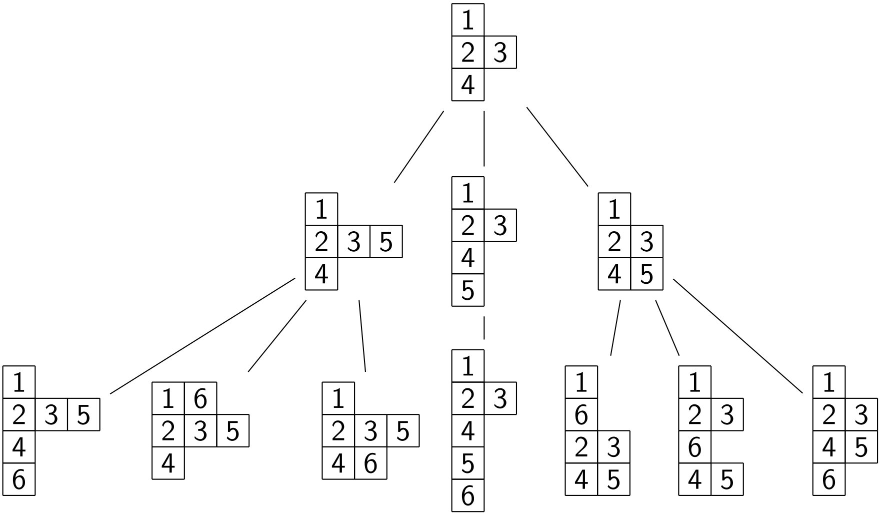
Create a set $\mathcal{QO}(\alpha * \lambda)$ in the following way:

1. place the entries $1, \ldots, |\alpha|$ into the diagram of α in reading order (left to right, starting with the top row),
2. for each i, $|\alpha| + 1 \leq |\alpha| + |\lambda|$, once $i - 1$ has been placed, follow the rules for the placement of i into the tableau:
    1. If $i - 1$ and i are in the same row of $S_{\alpha * \lambda}$, i must be placed in a column strictly right of $i - 1$ such that once i is placed at the end of a row, there is no row of the same length below it.
    2. If i is in the same column as y, $y < i$, in $S_{\alpha * \lambda}$, then i must be placed in a column weakly left of y such that once i is placed at the end of a row, there is no row of the same length below it.
3. Keep track of each possible placement of i by using a tree.
4. If no placement of i is possible, mark as a dead end and disregard.
5. The elements of $\mathcal{QO}(\alpha * \lambda)$ are the leaves of the tree which are not dead ends.

Theorem (N.)

Given α, λ, and β, the number of tableaux in $\mathcal{QO}(\alpha * \lambda)$ of shape β is the number of Littlewood-Richardson composition tableaux of shape β/α with content λ.

The tree generating elements of $\mathcal{QO}((1,2,1)*(1,1))$



$$\mathcal{C}_{(1,2,1)}s_{(1,1)} = \mathcal{C}_{(1,1,3,1)} + \mathcal{C}_{(1,3,2)} + \mathcal{C}_{(2,3,1)}$$
$$+ \mathcal{C}_{(2,2,1,1)} + \mathcal{C}_{(2,1,2,1)} + \mathcal{C}_{(1,2,2,1)} + \mathcal{C}_{(1,1,1,2,1)}$$

Other Facts

- There is a row-strict version $\mathcal{R}_\alpha$ of the quasisymmetric Schur functions. These also refine the Schur functions:

$$s_{\lambda'} = \sum_{\widetilde{\alpha}=\lambda} \mathcal{R}_\alpha.$$

- There is another version of the Remmel-Whitney rule that applies to the $\mathcal{R}_\alpha$. It is not the same as the rule for $\mathcal{C}_\alpha$.
- The main differences between the rule for quasisymmetric Schur functions and Schur functions are
  - the manner in which new rows are created,
  - adherence to triple rules, and
  - the possibility of a "dead end" or leaf that must be disregarded.

Future Work

- Adapt for skew quasisymmetric Schur functions.
- Adapt for $\mathcal{C}_\alpha \mathcal{C}_\beta$. This will be a particular challenge since $\mathcal{C}_\alpha \mathcal{C}_\beta$ does not necessarily expand positively in the quasisymmetric Schur basis. An appropriate adaptation will require both incorporating signs and having a rule that may remove/rearrange boxes.
- Look at products like $\mathcal{C}_\alpha \mathcal{R}_\beta$.