Reverse Plane Partitions and Quiver Representations

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Algebraic Combinatorixx 2

BIRS

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Outline

- Reverse Plane Partitions and Rim Hooks
- Quiver Representations
  (Exposition from R. Schiffler's book)
- Our problem

Reverse Plane Partitions and Rim Hooks

A **reverse plane partition** is a filling of a Young diagram with non-negative integers such that

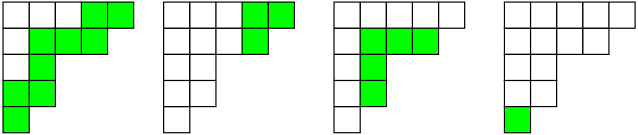
- entries in rows are weakly increasing, and
- entries in columns are weakly increasing.

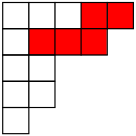
0	0	2	2
0	3		
6			

1	2	2
2	2	4
2	6	

Reverse Plane Partitions and Rim Hooks

A **rim hook** is a connected sequence of cells along the southeast border of a Young diagram such that its removal leaves a smaller Young diagram.

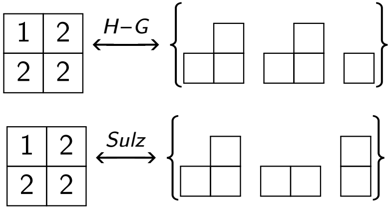




Reverse Plane Partitions and Rim Hooks

Theorem (Hillman-Grassl 1976, Pak 2001, Sulzgruber 2016)

There is a bijection between reverse plane partitions of shape λ and multisets of rim hooks of λ.



This result has many enumerative applications. (Gansner (1981), Morales-Pak-Panova (2016))

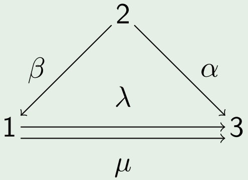
Quiver Representations

Definition

A **quiver** Q is a directed graph on a set of vertices Q_0 with a set of arrows Q_1.

Example

$Q =$



$$Q_0 = \{1, 2, 3\} \text{ and } Q_1 = \{\alpha, \beta, \lambda, \mu\}$$

Quiver Representations

Quiver: a case for carrying or holding arrows

Quiver Representations

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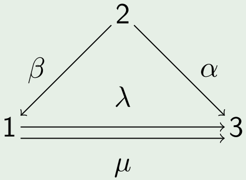
$Q =$ 

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Quiver Representations

Definition

Let k be an algebraically closed field. A **representation** $M = (M_i, \phi_\alpha)_{i \in Q_0, \alpha \in q_1}$ **of a quiver** Q consists of

- a k-vector space M_i at each vertex of Q
- a linear map ϕ_α for each arrow of Q, where $\phi_\alpha : \text{tail}(\alpha) \to \text{head}(\alpha)$

Example

Let $Q = 1 \longrightarrow 2$. Below is a representation of Q.

$$k \xrightarrow{\quad 1 \quad} k$$

Quiver Representations

Example

Let $Q = 1 \longrightarrow 2$. Some representations of Q are shown below.

$$k \xrightarrow{\quad 1 \quad} k$$

$$k \xrightarrow{\quad 0 \quad} 0$$

$$k^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}} k^3$$

Quiver Representations

A quiver representation M is **indecomposable** if $M \neq 0$ and M cannot be written as $M \cong M_1 \oplus M_2$, where M_1 and M_2 are nonzero quiver representations.

Theorem (Krull-Schmidt)

Let Q be a quiver and let M be a representation of Q. Then

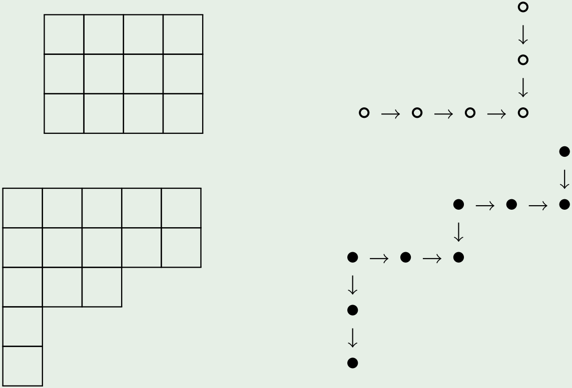
$$M \cong M_1 \oplus M_2 \oplus \cdots \oplus M_t,$$

where the M_i are indecomposable quiver representations and are unique up to order.

Indecomposable Quiver Representations and Rim Hooks

Starting with a Young diagram, we can make a (type A) quiver.

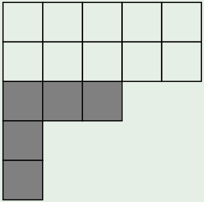
Example



Indecomposable Quiver Representations and Rim Hooks

Rim hooks of the Young diagram correspond to certain indecomposable representations of the quiver (the indecomposables that inject into the sincere indecomposable representation).

Example



$$\begin{array}{ccccccccc} & & & & & & & & 0 \\ & & & & & & & & \downarrow \\ & & & & 0 & \to & 0 & \to & 0 \\ & & & & \downarrow & & & & \\ k & \to & k & \to & k & & & & \\ \downarrow & & & & & & & & \\ k & & & & & & & & \\ \downarrow & & & & & & & & \\ k & & & & & & & & \end{array}$$

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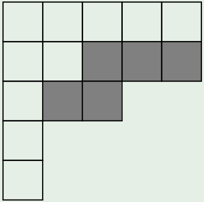

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Indecomposable Quiver Representations and Rim Hooks

Recall:

Theorem (Hillman-Grassl 1976, Pak 2001, Sulzgruber 2016)

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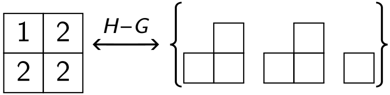

This gives us a bijection between certain quiver representations and reverse plane partitions.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 2 \\ \hline \end{array} \overset{H-G}{\longleftrightarrow} \left(\begin{array}{ccc} & & k \\ & & \downarrow \\ k & \longrightarrow & k \end{array} \oplus \begin{array}{ccc} & & k \\ & & \downarrow \\ k & \longrightarrow & k \end{array} \oplus \begin{array}{ccc} & & 0 \\ & & \downarrow \\ 0 & \longrightarrow & k \end{array} \right)$$

Obtaining a Representation from a Reverse Plane Partition

We would like to give these bijections algebraic meaning in the setting of quiver representations. For the remainder of the talk, we will use the bijection described by Pak.

Goal

Given a reverse plane partition P, describe a way to directly obtain a quiver representation such that the decomposition of this representation agrees with the decomposition of P into rim hooks.

On our way to making this precise, we need a few notions.

Obtaining a Representation from a Reverse Plane Partition

- Suppose we have a quiver with a vector space at each vertex, and in addition we have a nilpotent operator at each vertex.

$$\begin{array}{ccccc} k & \longrightarrow & k^3 & \longleftarrow & k^2 \\ N_1 & & N_2 & & N_3 \end{array}$$

Obtaining a Representation from a Reverse Plane Partition

- Suppose we have a quiver with a vector space at each vertex, and in addition we have a nilpotent operator at each vertex.

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- Let M be a quiver representation using our quiver with the attached vector spaces. We say M is **compatible** with $(N_1, N_2, \ldots, N_k)$ if the following commutes:

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_{M_i}} & M_{i+1} \\ N_i \downarrow & & \downarrow N_{i+1} \\ M_i & \xrightarrow[\varphi_{M_{i+1}}]{} & M_{i+1} \end{array}$$

In other words, $N \in Hom(M, M)$ in the category of quiver reps.

Obtaining a Representation from a Reverse Plane Partition

Q: What does it mean to choose a **generic** representation M that is compatible with nilpotent linear operators $N = (N_1, \ldots, N_k)$?

Obtaining a Representation from a Reverse Plane Partition

Q: What does it mean to choose a **generic** representation M that is compatible with nilpotent linear operators $N = (N_1, \ldots, N_k)$?

- Take all representations compatible with N.
- This is a subvariety of the variety of representations.
- There is an action of $G = \prod_{i \in Q_0} GL_{d_i}(k)$ on representations.
- There is one G-orbit whose intersection with this subvariety is dense in the subvariety.

There is also a notion of a generic N.

Morally: There are no coincidences.

Obtaining a Representation from a Reverse Plane Partition

- Start with a reverse plane partition P.

1	2
1	2

Obtaining a Representation from a Reverse Plane Partition

- Start with a reverse plane partition P.

1	2
1	2

- Make the corresponding quiver Q and put a vector space on each vertex with dimension determined by the corresponding diagonal of P.

$$\begin{array}{ccc} & & k^2 \\ & & \downarrow \\ k & \longrightarrow & k^3 \end{array}$$

Obtaining a Representation from a Reverse Plane Partition

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$$k \longrightarrow k^3 \longleftarrow k^2$$

Obtaining a Representation from a Reverse Plane Partition

- Start with a reverse plane partition P.

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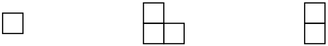
- Make the corresponding quiver Q and put a vector space on each vertex with dimension determined by the corresponding diagonal of P.
- Fix a generic nilpotent linear operator for each vector space, where the block sizes of the Jordan canonical form of the operator is determined by the corresponding diagonal of P.

$$k \longrightarrow k^3 \longleftarrow k^2$$



Obtaining a Representation from a Reverse Plane Partition

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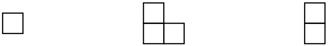
- Make the corresponding quiver Q and put a vector space on each vertex with dimension determined by the corresponding diagonal of P.
- Fix a generic nilpotent linear operator for each vector space, where the block sizes of the Jordan canonical form of the operator is determined by the corresponding diagonal of P.
- Let M be a generic representation on these vector spaces that is compatible with the nilpotent operators.

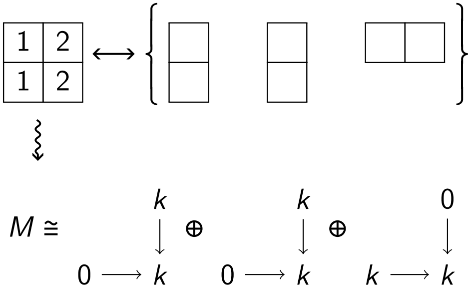
$$k \longrightarrow k^3 \longleftarrow k^2$$



Obtaining a Representation from a Reverse Plane Partition

Theorem (Garver-P.-Thomas)

The decomposition of M into indecomposable representations will agree with the decomposition of the reverse plane partition P into rim hooks.



Thank you!