On the growth of the Kronecker coefficients.

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Summary

What are the Kronecker coefficients?

- The representation theory of the general lineal group.
- The representation theory of the symmetric group.
- Symmetric functions.

The representation theory of the general lineal group

A representation of $GL(V)$ is a vector space W together with a group homomorphism

$$\rho : GL(V) \to GL(W).$$

The character of ρ is a function from $GL(V) \to \mathbb{C}$ that to any matrix A it associates the trace of $\rho(A)$.

We assume that our represention are polynomial.

Defining representation

Let $\{e_1, e_2, \cdots, e_d\}$ a eigenbasis for A , that we can assume to be diagonalizable.

$$Ae_k = \alpha_k e_k$$

The identity $\phi(A) = A$ defines a representation $GL(V) \to GL(V)$ of degree d.

Its character is just the trace of A, i.e., the sum of the eigenvalues of A:

$$ch(V) = s_1(\alpha_1, \alpha_2, \cdots, \alpha_d) = \alpha_1 + \alpha_2 + \cdots + \alpha_d$$

Symmetric Powers, $Sym^k V$

There is a representation of $GL(V)$ on $Sym^k(V)$ induced by the diagonal action of $GL(V)$ on $Sym^k V$.

$$Sym^k : GL(V) \to GL(Sym^k V))$$

A basis for $Sym^k V$ is given by the symmetric products

$$e_{i_1} \odot e_{i_2} \odot \cdots \odot e_{i_k}$$

with $i_1 \leq i_2, \cdots \leq i_k$.

The character of $Sym^k V$

Then,

$$A(e_{i_1} \odot e_{i_2} \odot \cdots \odot e_{i_k}) = A(e_{i_1}) \odot A(e_{i_2}) \odot \cdots \odot A(e_{i_k})$$
$$= (\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}) e_{i_1} \odot e_{i_2} \odot \cdots \odot e_{i_k}$$

The trace of this homomorphism is the sum of all different monomials $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}$ of degree k:

$$ch(Sym^k V) = s_{(k)}(\alpha_1, \alpha_2, \cdots, \alpha_d)$$

The exterior powers, $\bigwedge^k V$, with $k \leq \dim V$

Let A act on $\bigwedge^k V$ diagonally. This defines a representation

$$\bigwedge^k V : GL(V) \to GL(\bigwedge^k (V))$$

$$\begin{aligned} A(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}) &= A(e_{i_1}) \wedge A(e_{i_2}) \wedge \cdots \wedge A(e_{i_k}) \\ &= (\alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_k}) e_{i_1} \odot e_{i_2} \wedge \cdots \wedge e_{i_k} \end{aligned}$$

with strictly increasing subindices.

The character of the representation $\bigwedge^k V$ is

$$ch(\bigwedge^k V) = s_{(1^k)}(\alpha_1, \alpha_2, \cdots, \alpha_d)$$

Weyl's construction

- To any partition λ of length $\leq \dim(V)$ Weyl associated an irrep of $GL(V)$ inside of $V^{\otimes}$ using Young symmetrizers.
- A basis of W^{λ} is labelled by all semistandard filling of λ.
- The degree of the representation $W^{\lambda}(V)$ is the number of semistandard filling of λ with entries $1, 2, \cdots, \dim(V)$.

The restriction of representations

The product $GL(V) \times GL(W)$ can be embedded into $GL(V \otimes W)$:

Let $A \in GL(V)$ and $B \in GL(W)$, (A, B) acts on $V \otimes W$:

$$(A, B)(v \otimes w) = Av \otimes Bw.$$

That is,

$$GL(V) \times GL(W) \to GL(V \otimes W) \to W^{\lambda}(V \otimes W)$$

The restriction of $W^{\lambda}(V \otimes W)$ to $GL(V) \times GL(W)$ defines a (usually reducible) representation of $GL(V) \times GL(W)$.

The Kronecker coefficients

The restriction of representations is usually reducible

$$W^{\lambda}(V \otimes W) \downarrow^{GL(V \otimes W)}_{GL(V) \times GL(W)} \cong \bigoplus g_{\mu,\nu,\lambda} W^{\mu}(V) \otimes W^{\nu}(W)$$

The $g_{\lambda,\mu,\nu}$ are the Kronecker coefficients.

Being multiplicities are non-negative integers.

The Character of $V \otimes W$

Let $\{e_1, e_2, \cdots e_{d_1}\}$ and $\{f_1, f_2, \cdots f_{d_2}\}$ be eigenbases for A and B, with eigenvalues α_i and β_j.

Then $e_i \otimes f_j$ is an eigenbasis for $V \otimes W$ with eigenvalues $\alpha_i \beta_j$

$$(A, B)(e_i \otimes f_j) = A(e_i) \otimes B(f_j) = \alpha_i \beta_j (e_i \otimes f_j)$$

Therefore,

$$ch(V \otimes W) = s_1(\alpha_1\beta_1, \alpha_1\beta_2 \cdots, \alpha_{d_1}\beta_{d_2})$$

$ch(W^{\lambda}(V \otimes W))$

Since $ch(W^{\lambda}(V \otimes W))$ is the Schur function s_{λ} evaluated at the eigenvalues for the representation $V \otimes W$

$$ch(W^{\lambda}(V \otimes W)) = s_{\lambda}(\alpha_1\beta_1, \alpha_1\beta_2 \cdots, \alpha_{d_1}\beta_{d_2})$$

Summary

The isormorphism of representations

$$W^{\lambda}(V \otimes W) \downarrow^{GL(V \otimes W)}_{GL(V) \times GL(W)} \cong \bigoplus g_{\mu,\nu,\lambda} W^{\mu}(V) \otimes W^{\nu}(W)$$

translates into the identity of symmetric functions:

$$s_{\lambda}(\alpha_1\beta_1, \alpha_1\beta_2 \cdots, \alpha_{d_1}\beta_{d_2}) = \sum_{\mu,\nu} g_{\mu,\nu,\lambda}\, s_{\mu}(\alpha_1, \cdots, \alpha_{d_1}) s_{\nu}(\beta_1, \cdots, \beta_{d_2})$$

The product of two alphabets

If $A = \alpha_1 + \cdots + \alpha_{d_1}$ and $B = \beta_1 + \cdots + \beta_{d_2}$ then

$$AB = \alpha_1\beta_1 + \alpha_1\beta_2 \cdots + \alpha_{d_1}\beta_{d_2}$$

Therefore,

$$s_\lambda(\alpha_1\beta_1, \alpha_1\beta_2 \cdots, \alpha_{d_1}\beta_{d_2}) = \sum_{\mu,\nu} g_{\mu,\nu,\lambda}\, s_\mu(\alpha_1, \cdots, \alpha_{d_1}) s_\nu(\beta_1, \cdots, \beta_{d_2})$$

Becomes

$$s_\lambda[AB] = \sum_{\mu,\nu} g_{\mu,\nu,\lambda}\, s_\mu[A] s_\nu[B]$$

The stretched Kronecker quasi–polynomial $Q_{\lambda,\mu,\nu}(t)$

Let λ, μ and ν be three partitions of n.

The stretched Kronecker quasi–polynomial $Q_{\lambda,\mu,\nu}(t)$ is defined as

$$Q_{\lambda,\mu,\nu}(t) = g_{t\lambda,t\mu,t\nu}$$

Thm [Meinrenken-Sjamaar 1999, Mulmuley 2007]
The function $Q_{\lambda,\mu,\nu}(t)$ is indeed a quasi–polynomial.

Some examples of Kronecker quasi-polynomials

- For $\lambda = (1,1), \mu = (1,1), \nu = (1,1)$

$$g_{\lambda,\mu,\nu} = 0, \qquad Q_{\lambda,\mu,\nu}(t) = \begin{cases} 1, & \text{if } t \equiv 0 \mod 2 \\ 0, & \text{if } t \equiv 1 \mod 2 \end{cases}$$

- For $\lambda = (2,1), \mu = (2,1), \nu = (2,1)$

$$g_{\lambda,\mu,\nu} = 2, \quad Q_{\lambda,\mu,\nu}(t) = \begin{cases} (t+2)/2, & \text{if } t \equiv 0 \mod 2 \\ (t+1)/2, & \text{if } t \equiv 1 \mod 2 \end{cases}$$

Lineal growth

A related result

Thm [Briand, Rattam, R]

If the three indexing partitions have long first parts, and we add t cells to each of their first two rows,

then, the Kronecker coefficients are described by a linear quasipolynomial of period 2 on t.

The remaining rows

Thm [Colmenarejo-R:]

The reduced Kronecker coefficient indexed by $(k), (k^a), (k^a)$ count the number of plane partitions of k fitting inside a $2 \times a$ rectangle.

The corresponding quasi-polynomial has degree $2a - 1$ and period $lcm(1, 2, \cdots, a+1)$.

A cubic quasi–polynomial

Thm [Colmenarejo-R:]

As an illustration, we include the following example that shows that the coefficients $\overline{g}^{(k)}_{(k^2),(k^2)}$ are given by the following quasipolynomial of degree 3 and period 6:

$$\overline{g}^{(k)}_{(k^2),(k^2)} = \begin{cases} 1/72k^3 + 1/6k^2 + \ \ 2/3k + \ \ 1 & \text{if } k \equiv 0 \mod 6 \\ 1/72k^3 + 1/6k^2 + 13/24k + 5/18 & \text{if } k \equiv 1 \mod 6 \\ 1/72k^3 + 1/6k^2 + \ \ 2/3k + 8/9 & \text{if } k \equiv 2 \mod 6 \\ 1/72k^3 + 1/6k^2 + 13/24k + 1/2 & \text{if } k \equiv 3 \mod 6 \\ 1/72k^3 + 1/6k^2 + \ \ 2/3k + 7/9 & \text{if } k \equiv 4 \mod 6 \\ 1/72k^3 + 1/6k^2 + 13/24k + 7/18 & \text{if } k \equiv 5 \mod 6 \end{cases}$$

Why is this interesting?

- Asymptotic of the Kronecker coefficients.

- This quasi–polynomial gives us information about the kind of objects that can be counted by the Kronecker coefficients.

An interesting example of Kronecker quasi–polynomial

Let $\lambda = (6,6)$, $\mu = (7,5)$, and $\nu = (6,4,2)$ be partitions.

Briand, Orellana, R showed that Mulmuley's saturation hypothesis does not hold.

For even values of t

$$Q_{\lambda,\mu,\nu} = \frac{1}{2}(t+2).$$

whereas for odd values of t

$$Q_{\lambda,\mu,\nu} = \frac{1}{2}(t+2).$$

Ron King's observation

In 2009, in Sevilla, Ron King observed that for $\lambda = (6,6)$, $\mu = (7,5)$, and $\nu = (6,4,2)$

$$Q_{\lambda,\mu,\nu}(m) \not\geq (-1)^m Q_{\lambda,\mu,\nu}(-m)$$

Ehrhart reciprocity fails.

$Q_{\lambda,\mu,\nu}$ can NOT count the number of integral points on any dilated rational polytope.

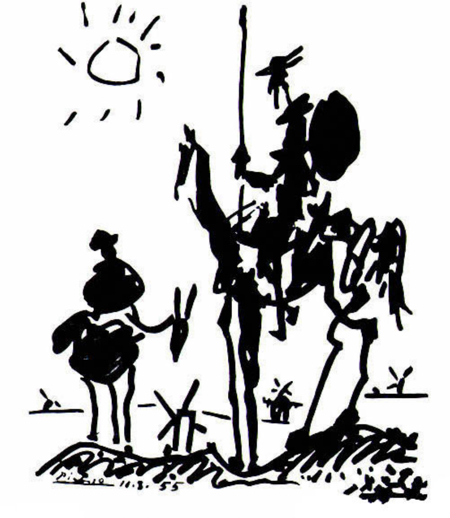
The work of Baldoni and Vergnes

On their paper Computations of dilated Kronecker coefficients Baldoni and Vergnes found an algorithm that computes the dilated Kronecker coefficients, and implemented it on Maple.

For partitions of bounded length it runs in polynomial time.

There are also plenty of interesting examples.

Only Don Quijote would be find it realistic to try to describe combinatorial formulas for the Kronecker quasi-polynomials.



Garsia, Wallach, Xin, Zabrocki:

$$s_{(d,d),(d,d),\lambda} = ((\text{all parts odd or all parts even}))$$

Brown, van Willigenburg, Zabrocki:

$$g_{(d,d),(d+k,d-k),\lambda} = 1\{ \text{ if } k \equiv \lambda_2 \, (mod\, 2) \text{ and } \lambda_2 \geq k$$

and zero otherwise

Ron King's conjectures

In his Sevilla talk, Ron King proposed a series of conjectures concerning the Kronecher quasi–polynomial $Q_{(n-1,1),\mu,\mu}$

$$Q^{\nu}_{\lambda\mu}(t) \textbf{ with } \lambda = (n-1,1) \textbf{ and } \mu = \nu$$

Cases with $a > 0$, $a > b > 0$ and $a > b > c > 0$ as appropriate for $t = 0, 1, \ldots, 12$

$\mu = \nu$	0	1	2	3	4	5	6	7	8	9	10	11	12
a	1	0	0	0	0	0	0	0	0	0	0	0	0
aa	1	0	1	0	1	0	1	0	1	0	1	0	1
ab	1	1	2	2	3	3	4	4	5	5	6	6	7
aaa	1	0	1	1	1	1	2	1	2	2	2	2	3
aab, abb	1	1	3	4	7	9	14	17	24	29	38	45	57
abc	1	2	6	12	24	42	72	114	177	262	380	534	738

Conjecture The results are independent of a, b, c

(taken from his slides)

Ron King's conjectures

$Q^{\nu}_{\lambda\mu}(t)$ **with** $\lambda = (n-1,1)$ **and** $\mu = \nu$

Cases with $a > b > c > d > e > 0$ for $t = 0, 1, \ldots, 10$

$\mu = \nu$	0	1	2	3	4	5	6	7	8	9	10
$aaaaa$	1	0	1	1	2	2	3	3	5	5	7
$abbbb$	1	1	3	5	10	15	26	38	60	85	125
$aabbb$	1	1	4	7	16	27	54	88	158	253	421
$abccc$	1	2	7	17	42	91	196				
$abbcc$	1	2	8	20	55	128	304				
$abcdd$	1	3	13	42	133	378	1029				
$abcde$	1	4	20	80	300	1020					

Conjecture The results are independent of a, b, c, d, e

The cases $abbbb$ and $aaaab$ are identical, etc.

(taken from his slides)

Ron King's conjectures

$Q^{\nu}_{\lambda\mu}(t)$ **with** $\lambda = (n-1,1)$ **and** $\mu = \nu$

Cases with $a > b > c > d > 0$ for $t = 0, 1, \ldots, 10$

$\mu = \nu$	0	1	2	3	4	5	6	7	8	9	10
$aaaa$	1	0	1	1	2	1	3	2	4	3	5
$abbb$	1	1	3	5	9	13	22	30	45	61	85
$aabb$	1	1	4	6	14	20	40	56	98	136	218
$abcc$	1	2	7	16	38	77	157	291	533	922	1566
$abcd$	1	3	12	36	102	258	616	1368	2892	5812	11220

Conjecture The results are independent of a, b, c, d

The cases $abbb$ and $aaab$ are identical

The cases $abcc$, $abbc$ and $aabc$ are identical

(taken from his slides)

A combinatorial proof for Ron King's conjectures

The starting point of our work will be a combinatorial interpretation for Kronecher quasi–polynomial $Q_{(n-1,1),\mu,\mu}$.

We will explore where does it takes us.