Permutation patterns and structures

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Permutations

S_n = permutations of $[n]$

There are many ways to write permutations: one-line notation, cycle notation, products of simple reflections, diagrams, etc.

Example. $4213 \in S_4$ is the permutation mapping $1 \mapsto 4$, $2 \mapsto 2$, $3 \mapsto 1$, and $4 \mapsto 3$.

Psst! S_n is the Coxeter group A_{n-1}!

Patterns

Let p and w be permutations.

$\boldsymbol{p \prec w}$: w has a **p-pattern** if a substring of the one-line notation for w has the same relative order as the one-line notation for p. Otherwise w **avoids** p.

Example. The permutation 45213 contains two copies of the pattern 321, and avoids the pattern 132.

Pattern variations

Patterns are trendy!

There are many variations on pattern avoidance/containment:

barred patterns
vincular patterns
bivincular patterns
mesh patterns
etc.

Pattern intrigue

Patterns are interesting!

There are two main areas of research:

Enumeration: How many permutations in S_n avoid the pattern p?

Characterization: Is p-avoidance equivalent to something else?

Psst! You can also do this with signed patterns and with other avoidance flavors!

Sampler of results

Thm. [Simion-Schmidt] Equally many $w \in S_n$ avoid 132 as 123.

In fact, this is true for avoiding any $p \in S_3$.

Thm. [Stankova] Enumerating the avoidance of $p \in S_4$ depends only on which of three categories p lies in.

Thm. [Knuth] A permutation avoids 231 iff the permutation is **stack-sortable**.

Thm. [Billey-Jockusch-Stanley] A permutation avoids 321 iff it is **fully commutative**.

Psst! Check out the Database of Permutation Pattern Avoidance!

Time to develop a handy-dandy tool

First, some background ...

The **simple reflections** on $[n]$ are the involutions:

$$\boldsymbol{s_i} = 1 \cdots (i-1)\boldsymbol{(i+1)i}(i+2) \cdots n$$

These generate S_n and satisfy the relations:

$s_i s_j = s_j s_i$ for $|i-j| > 1$ **(commutation)**

$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i \in [n-2]$ **(braid)**

$s_i w$ swaps the positions of the values i and $i+1$ in w

ws_i swaps the values in the positions i and $i+1$ in w

Reduced words

The **length** of w is the least $\boldsymbol{\ell} = \ell(w)$ for which $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$.

The string of subscripts $\boldsymbol{i_1 \cdots i_\ell}$ is a **reduced word** of w.

$\boldsymbol{R(w)}$ = set of reduced words of w.

Example. $R(4213) = \{1321, 3121, 3212\}$

A **factor** is a consecutive substring in a reduced word.

Jackpot!

Thm. Elements of $R(p)$ appear as **shifted isolated** factors in elements of $R(w)$ iff $\boldsymbol{p^+ \ll w}$.

When p is vexillary (i.e., p avoids 2143), $\boldsymbol{p^+ \ll w}$ becomes $\boldsymbol{p \prec w}$.

In other words, $p \prec w \iff$ reduced words for p "appear in" reduced words for w.

Why is this good?

1. We can now use pattern techniques to prove things about reduced words (and Bruhat order, etc.)!

2. We can now use reduced word techniques to prove things about pattern containment and avoidance!

Sampler of results

Thm. If $p \prec w$ then $|R(p)| \leq |R(w)|$.

Thm. If $p \prec w$ and $|R(p)| > 1$, then $|R(p)| = |R(w)|$ iff $\ell(p) = \ell(w)$.

$\boldsymbol{C(w)} = R(w)/ij \sim ji$ for $|i-j| > 1$, the **commutation classes**

Thm. If $p \prec w$, then $|C(p)| \leq |C(w)|$.

Thm. If $p \prec w$, then $|C(p)| = |C(w)|$ iff p and w have the same number of 321-patterns.

Sampler of results, cont.

B(w) $= \{v \in S_n : v \leq w$ in **strong Bruhat order**$\}$.

(Can be defined in terms of reduced words.)

If $B(w)$ is a boolean poset, then w is **boolean**.

Thm. w is boolean iff w avoids 321 and 3412.

Thm. $\#\{\text{boolean } w \in S_n : \ell(w) = k\} = \sum_{i=1}^{k} \binom{n-i}{k+1-i}\binom{k-1}{i-1}$

Thm. The cell complex whose face poset comes from those boolean elements is homotopy equivalent to a wedge of top-dimensional spheres.

Sampler of results, cont.

Thm. The number of 132-avoiding permutations of length ℓ is equal to the number of partitions of length ℓ.

Thm. The number of 132-avoiding permutations of length ℓ, in which $w(1) = k+1$, is equal to the number of partitions of length ℓ into exactly k parts.

Thm. The number, $X(\ell, d)$, of 132-avoiding permutations of length ℓ whose reduced words have d distinct letters is equal to the number of partitions of ℓ that fit into the staircase shape δ_{d+1} but not into δ_d.

Thm. $$\sum_{\substack{0 \le d < n \\ 0 \le \ell \le \binom{n}{2}}} X(\ell, d) = C_n$$

Sampler of results, cont.

Thm. The collection of elements in S_n that avoid a given pattern p is never an order ideal in the Bruhat order.

Consider convex centrally symmetric polygons with all sides of length 1.

Thm. Such a $2n$-gon can be tiled by such $2k$-gons iff $k \in \{2, n\}$.

Let's prove a few more!