Quasisymmetric Macdonald Polynomials

Sarah K Mason
Wake forest University

Algebraic Combinatorixx II
Banff International Research Station
May 15, 2017

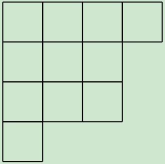
Outline

Partitions

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of a positive integer n is a weakly decreasing sequence of positive integers which sum to n. The Ferrers diagram of a partition (in "English notation") is a collection of left-justified boxes arranged into rows so that the i^{th} row contains λ_i boxes.

Example

$\lambda = (4, 3, 3, 1)$ is a partition of 11 with Ferrers diagram:

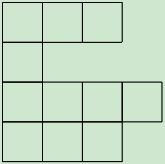


Compositions

A composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of a positive integer *n* is a sequence of positive integers which sum to *n*. The diagram of a composition (in “English notation”) is a collection of left-justified boxes arranged into rows so that the i^{th} row contains α_i boxes.

Example

$\lambda = (3, 1, 4, 3)$ is a composition of 11 with diagram:

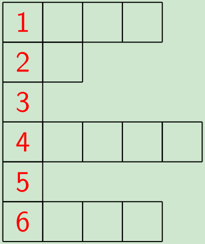


Weak Compositions

A <u>weak composition</u> $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ of a positive integer n is a sequence of *non-negative* integers which sum to n. The <u>diagram</u> of a weak composition (in "English notation") is a collection of left-justified boxes arranged into rows so that the i^{th} row contains γ_i boxes. (Basement = leftmost column, acts as index)

Example

$\lambda = (3, 1, 0, 4, 0, 3)$ is a composition of 11 with diagram:



Symmetric Functions Sym_n

A symmetric function $f(x_1, x_2, \ldots, x_n)$ in n commuting variables is a function which remains the same when the indices of the variables are permuted.

Example

$$f(x_1, x_2, x_3) = x_1^4 x_2 + x_1^4 x_3 + x_1 x_2^4 + x_1 x_3^4 + x_2^4 x_3 + x_2 x_3^4$$

(symmetric)

$$f(x_1, x_2, x_3) = x_1^3 x_2 + x_1 x_2^3 x_3^2 \qquad \text{(not symmetric)}$$

Bases for symmetric functions

- Monomial (symmetrize a monomial):
  $m_{2,1}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2$
- Elementary (no repeats):
  $e_{2,1} = (e_2)(e_1) = (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3)$
- complete Homogeneous (everything):
  $h_{2,1} = (h_2)(h_1) = (x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3)(x_1+x_2+x_3)$
- Power sum (raised to power):
  $p_{2,1} = (p_2)(p_1) = (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3)$

Schur functions

Definition

A reverse semi-standard Young tableau (SSYT) is a filling of a Ferrers diagram with positive integers so that the rows are weakly decreasing left to right and the columns are strictly decreasing top to bottom. The weight of a SSYT T is $x^T = \prod x_i^{a_i}$, where $a_i =$ the number of times the entry i appears in T.

Example

$$T = \begin{array}{|c|c|c|c|}\hline 6 & 6 & 3 & 2 \\ \hline 2 & 1 & 1 \\ \cline{1-3} 1 \\ \cline{1-1}\end{array}$$

is a semi-standard Yound tableau of shape $(4,3,1)$ and weight $x^T = x_1^2 x_2^2 x_3 x_6^2$.

Schur functions

Definition

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in SSYT(\lambda)} x^T,$$

where $SSYT(\lambda)$ is the set of all SSYT of shape λ.

Example

$$s_{2,1}(x_1, x_2, x_3) =$$

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$x_1^2x_2 + x_1^2x_3 + x_1x_2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_3^2 + x_2x_3^2$$

Schur functions...

- form a basis for all symmetric functions.
- are closely related to other symmetric function bases.
- correspond to characters of irr reps of GL_n.
- describe the cohomology of the Grassmannian.
- have many nice combinatorial properties.
- generalize to Macdonald polynomials $(P_\lambda(X; q, t))$.

Macdonald Polynomials (I. G. Macdonald, 1988)

$$P_\lambda(X_n; q, t) = m_\lambda(X_n) + \sum_{\mu<\lambda} u_{\lambda,\mu} m_\mu(X_n)$$

- Unique eigenfunction of a certain divided difference operator (generate a realization of an extended affine Hecke algebra)
- Orthogonal under a certain scalar product
- $P_\lambda(q, q) = s_\lambda$
- Contain other symmetric function bases as special cases
- Generalize zonal polynomials, Jack symmetric functions, etc
- Generalization of q-Selberg integral
- multivariable q-binomial theorem

Alphabet Soup

There are many different forms of Macdonald polynomials!

- Monic form: P_μ
- Integral form: J_μ
- Transformed integral forms: H_μ and $\tilde{H}_\mu$

Theorem (Haglund, Haiman, Loehr 2006)

Like Schur functions, Macdonald polynomials can be constructed combinatorially using fillings of diagrams.

$$F = \begin{array}{|c|c|c|c|}\hline 3 & 5 & 5 & 1 \\ \hline 5 & 1 & 4 & \multicolumn{1}{c}{} \\ \cline{1-3} 4 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \quad \text{contributes } x_1^2 x_3 x_4^3 x_5^3 q^2 t^4 \text{ to } \tilde{H}_\mu(X_9; q, t)$$

An arm and a leg (partition diagram)

- *arm*$(s)=\#$ boxes below s in same column as s
- $leg(s)=\#$ boxes to the right of s in same row as s

		s	ℓ	ℓ	ℓ
		a			
		a	

$$arm(s)=2, \quad leg(s)=3$$

An arm and a leg (composition diagram)

- $arm(s) = \#$ boxes in same column, below s, whose row is weakly shorter than the row containing s

  PLUS

  $\#$ boxes in column just left, whose row is above s and strictly shorter than the row containing s
- $leg(s) = \#$ boxes to the right of s in same row as s

1						
2		a				
3			s	ℓ		
4			a			
5						

$$arm(s) = 2, \quad leg(s) = 1$$

Non-symmetric Macdonald Polynomials

Macdonald polynomials break down into non-symmetric components:

$$P_\mu(X_n;q,t) = \prod_{s\in\mu}(1-q^{leg(s)+1}t^{arm(s)}) \sum_{\gamma^+=\mu} \frac{E_\gamma(X_n;q,t)}{\prod_{s\in\gamma}(1-q^{leg(s)+1}t^{arm(s)})}$$

which can then be specialized to $q=t=0$ to obtain a decomposition of the Schur functions:

$$P_\mu(X_n;0,0) = s_\mu(X_n) = \sum_{\tilde{\gamma}=\mu} E_\gamma(X_n;0,0),$$

where $\tilde{\gamma}$ is the rearrangement of γ into weakly decreasing order.

$$s_{21} = E_{210} + E_{201} + E_{021} + E_{120} + E_{102} + E_{012}$$

Quasisymmetric Functions

quasisymmetric functions in n variables ($QSym_n$)

$\sigma f(X) = f(X)$ for any shift σ of the nonzero exponents.
(Indexed by compositions.)

Examples ($QSym_3$)

- $x_1^2 + x_2^2 + x_3^2 = x_1^2 x_2^0 x_3^0 + x_1^0 x_2^2 x_3^0 + x_1^0 x_2^0 x_3^2$
- $x_1 x_2^3 + x_1 x_3^3 + x_2 x_3^3 = x_1^1 x_2^3 x_3^0 + x_1^1 x_2^0 x_3^3 + x_1^0 x_2^1 x_3^3$

Non-example

- $x_1^3 x_2 + x_1^3 x_3$ $\qquad \cancel{x_2^3 x_3}$

Quasisymmetric Schur functions

Semi-standard composition tableau (CT)

rows: weakly decreasing left to right

leftmost column: strictly increasing top to bottom

columns: $a \leq b \Rightarrow b > c$

$$\begin{array}{|c|c|}\hline c & a \\ \hline \multicolumn{1}{c}{} & \vdots \\ \cline{2-2} \multicolumn{1}{c|}{} & b \\ \cline{2-2}\end{array}$$

$$F = \begin{array}{|c|c|c|cc}\hline 3 & 2 & 1 & \multicolumn{2}{c}{} \\ \hline 6 & 6 & 3 & \multicolumn{2}{c}{} \\ \hline 7 & 4 & \multicolumn{3}{c}{} \\ \hline 9 & 8 & 8 & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

$$x^F = x_1 x_2 x_3^3 x_4 x_6^3 x_7 x_8^2 x_9, \qquad F \in CT(3,3,2,5)$$

Quasisymmetric Schur functions

Semi-standard composition tableau (CT)

rows: weakly decreasing left to right

leftmost column: strictly increasing top to bottom

columns: $a \leq b \Rightarrow b > c$

$$\begin{array}{|c|c|} \hline c & a \\ \hline \end{array}$$

$$\begin{array}{c|c|} \cline{2-2} & b \\ \cline{2-2} \end{array}$$

$$F = \begin{array}{|c|c|c|c|c|} \hline 3 & \color{red}{2} & \color{red}{1} \\ \cline{1-3} 6 & 6 & 3 \\ \cline{1-3} 7 & 4 \\ \hline 9 & 8 & \color{red}{8} & 6 & 3 \\ \hline \end{array}$$

$$x^F = x_1 x_2 x_3^3 x_4 x_6^3 x_7 x_8^2 x_9, \qquad F \in CT(3,3,2,5)$$

Quasisymmetric Schur functions

Semi-standard composition tableau (CT)

rows: weakly decreasing left to right

leftmost column: strictly increasing top to bottom

columns: $a \leq b \Rightarrow b > c$

$$\begin{array}{|c|c|} \hline c & a \\ \hline \end{array}$$
$$\begin{array}{cc} & \\ & \boxed{b} \end{array}$$

$$F = \begin{array}{|c|c|c|c|c|} \hline 3 & 2 & 1 \\ \cline{1-3} 6 & 6 & 3 \\ \cline{1-3} 7 & 4 \\ \hline 9 & 8 & 8 & 6 & 3 \\ \hline \end{array}$$

$$x^F = x_1 x_2 x_3^3 x_4 x_6^3 x_7 x_8^2 x_9, \qquad F \in CT(3,3,2,5)$$

Quasisymmetric Schur functions

Quasisymmetric Schurs

$$\mathcal{QS}_\gamma(x_1, \cdots, x_n) = \sum_{F \in CT(\gamma)} x^F$$

$$\mathcal{QS}_{2,1,3}(x_1, x_2, x_3) = x_1^2 x_2^2 x_3^2 + x_1^2 x_2 x_3^3$$

$$\begin{array}{|c|c|c} \hline 1 & 1 & \multicolumn{1}{|c}{} \\ \hline 2 & \multicolumn{1}{|c}{} & \\ \hline 3 & 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \qquad \begin{array}{|c|c|c} \hline 1 & 1 & \multicolumn{1}{|c}{} \\ \hline 2 & \multicolumn{1}{|c}{} & \\ \hline 3 & 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

$$s_\lambda = \sum_{\tilde{\alpha} = \lambda} QS_\alpha, \quad QS_\alpha = \sum_{\gamma^+ = \alpha} E_\gamma(X_n; 0, 0)$$

$$s_{21} = QS_{21} + QS_{12}, \qquad QS_{21} = E_{210} + E_{201} + E_{021}$$

Quasisymmetric Macdonald Polynomials

$$\begin{array}{ccccc}
\text{symmetric} & & \text{quasisymmetric} & & \text{non-symmetric} \\
P_\mu(X_n; q, t) & = & \displaystyle\sum_{\tilde{\alpha}=\mu} {\color{red}???}_\alpha & = & \displaystyle\sum_{\tilde{\gamma}=\mu} E_\gamma(X_n; q, t) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \quad q = t = 0 \\
s_\mu(X_n) & = & \displaystyle\sum_{\tilde{\alpha}=\mu} QS_\alpha & = & \displaystyle\sum_{\tilde{\gamma}=\mu} E_\gamma(X_n; 0, 0)
\end{array}$$

partitions	compositions	weak compositions
$(2,1)$	$(2,1),(1,2)$	$(2,1,0),(2,0,1),(0,2,1),$ $(1,2,0),(1,0,2),(0,1,2)$

What we are looking for:

Polynomial $I_\alpha(X_n; q, t)$, indexed by compositions α

- quasisymmetric in $x_1, x_2, \ldots, x_n$
- specialize to quasisymmetric Schurs when $q = t = 0$
- sum of nonsymmetric Macdonald polynomials
- described through fillings of composition diagrams
- orthogonal under a certain inner product

Initial Reading List

- Haglund, J., Haiman, M., and Loehr, N. A combinatorial formula for nonsymmetric Macdonald polynomials. *Amer. J. Math.* 130 (2008) 2:359-383.
- Haglund, J., Luoto, K., Mason, S., and van Willigenburg, S. Quasisymmetric Schur functions. *J. Combin. Theory Ser. A*, 118 (2011) 2:463-490.
- Marshall, D. Symmetric and Non-symmetric Macdonald Polynomials. *Annals of Combinatorics*, 3 (1999) 385-415.