A Connection for Born Geometry and its Application to DFT

String and M-theory Geometries

Banff International Research Station

January 26, 2017

Felix J. Rudolph

Ludwig-Maximilian-Universtität München

to appear

17xx.xxxxx

with L. Freidel and D. Svoboda

Key Points

- What is Born Geometry? (idea of dynamical phase space)
- Connections for various geometries
- Torsion and integrability
- Relevance to Double Field Theory

Outline

Born Geometry

Born Reciprocity

In Quantum Mechanics

- symmetry between spacetime and momentum space
- freedom to choose a basis of states

In General Relativity

- this symmetry is broken:

spacetime is curved	energy-momentum space is flat

Max Born (1935):

"To unify QM and GR need momentum space to be curved"

Born Geometry

[Freidel, Leigh, Minic]

Classical Mechanics

- (almost) symplectic structure ω on phase space $\mathcal{P}$ $Sp(2d)$

Quantum Mechanics

- complex structure I:

$$x \to p, \quad p \to -x \quad \text{with} \quad I^2 = -1$$

- compatibility: $I^T \omega I = \omega$
- defines a metric on phase space $\mathcal{H} = \omega I$ $O(2d, 0)$
- "quantum" or "generalized" metric

Born Geometry

To split phase space into spacetime and momentum space

- bi-Lagrangian (real) structure K: $T\mathcal{P} = L \oplus \tilde{L}$

$$K\big|_L = +1, \quad K\big|_{\tilde{L}} = -1 \quad \text{with} \quad K^2 = +1$$

- compatibility: $K^T \omega K = -\omega$
- defines another metric on phase space $\eta = \omega K$ $\qquad O(d,d)$
- "polarization" or "neutral" metric
- spacetime is maximal null subspace w.r.t. η

$$\mathcal{H}\big|_L = g$$

Born Geometry

The Born Geometry $(\mathcal{P}; \eta, \omega, \mathcal{H})$ unifies

- symplectic structure of classical mechanics
- complex strucuture of quantum mechanics
- real structure of general realitivity

Quantum gravity needs a dynamical phase space

String theory provides a realization of these concepts and geometric structure

String Theory

Tseytlin Action on phase space with $X = (x/\lambda, y/\epsilon)$

$$S = \frac{1}{2}\int \mathrm{d}\tau\mathrm{d}\sigma\left[(\eta_{AB} + \omega_{AB})\partial_\tau X^A \partial_\sigma X^B - \mathcal{H}_{AB}\partial_\sigma X^A \partial_\sigma X^B\right]$$

- including topological term [Giveon, Rocek; Hull]

String Theory

- chiral structure: $J = \eta^{-1}\mathcal{H}$
- T-duality on target space: $X \to J(X)$
- ω and K not required, but present

Para-quaternionic Manifold

Born Geometry $(\mathcal{P}; \eta, \omega, \mathcal{H}) \longrightarrow$ para-quaternions (I, J, K)

- complex structure $I = \mathcal{H}^{-1}\omega \qquad (I^2 = -1)$
- chiral structure $J = \eta^{-1}\mathcal{H} \qquad (J^2 = +1)$
- real structure $K = \eta^{-1}\omega \qquad (K^2 = +1)$

All mutually anti-commute

$$\begin{aligned} I &= JK = -KJ, \\ J &= IK = -KI, \\ K &= JI = -IJ, \end{aligned} \qquad IJK = -1$$

Integrability

Almost bi-Lagrangian structure K

- Splitting into Lagrangian distributions: $T\mathcal{P} = L \oplus \tilde{L}$
- $K^T \omega K = -\omega \quad \to \quad$ Lagrangian eigenspace $\omega\big|_L = \omega\big|_{\tilde{L}} = 0$
- $K^T \eta K = -\eta \quad \to \quad$ Null eigenspace $\eta\big|_L = \eta\big|_{\tilde{L}} = 0$

If K integrable:

- $[L, L] \subset L$ and $[\tilde{L}, \tilde{L}] \subset \tilde{L}$
- Induces a polarization
- Darboux coordinates $(x, \tilde{x})$ spanning L and $\tilde{L}$

Projectors of $(3,0)$-tensors

Lagrangian Subspaces (Polarizations): $T\mathcal{P} = L \oplus \tilde{L}$

- $(\mathcal{K}^{\pm})^2 = \mathcal{K}^{\pm}, \quad \mathcal{K}^{+}\mathcal{K}^{-} = 0$

$$4\mathcal{K}^{\pm}N(X,Y,Z) := N(X,Y,Z) + N(K(X),K(Y),Z) \pm N(X,K(Y),K(Z)) \pm N(K(X),Y,K(Z))$$

Chiral subspaces: $T\mathcal{P} = C_{+} \oplus C_{-}$

- $(\mathcal{J}^{\pm})^2 = \mathcal{J}^{\pm}, \quad \mathcal{J}^{+}\mathcal{J}^{-} = 0$

$$4\mathcal{J}^{\pm}N(X,Y,Z) := N(X,Y,Z) + N(J(X),J(Y),Z) \pm N(X,J(Y),J(Z)) \pm N(J(X),Y,J(Z))$$

Torsion

For a connection $\nabla = \partial + \Gamma$

Usual Torsion

- $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$
- $T_{ij}{}^k = \Gamma_{ij}{}^k - \Gamma_{ji}{}^k$

Generalized Torsion

- $\mathcal{T}(X,Y) = \mathcal{L}_X^{\nabla} Y - \mathcal{L}_X^{\partial} Y$
- $\mathcal{T}_{ij}{}^k = \Gamma_{ij}{}^k - \Gamma_{ji}{}^k - \Gamma^k{}_{ji}$
- $\mathcal{T} \in \Gamma(\Lambda^2(\mathcal{P}) \otimes \mathfrak{X}(\mathcal{P}))$

Contorsion

For any metric-compatible connection ∇ and Levi-Civita connection $\mathring{\nabla}$:

Contorsion tensor Ω

- $\nabla = \mathring{\nabla} + \Omega$ or $\Gamma = \mathring{\Gamma} + \Omega$
- $\Omega_{ijk} = \frac{1}{2}(T_{ijk} - T_{jki} + T_{kij})$
- $\mathcal{T}_{ijk} = \Omega_{ijk} + \Omega_{jki} + \Omega_{kij} = \frac{1}{2}(T_{ijk} + T_{jki} + T_{kij})$

The Nijenhuis Tensor

The Nijenhuis Tensor of a tangent bundle structure A

$$N_A \in \Gamma(\Lambda^2(\mathcal{P}) \otimes \mathfrak{X}(\mathcal{P}))$$

$$N_A(X,Y) = A\big([A(X),Y]+[X,A(Y)]\big)-[A(X),A(Y)]-A^2[X,Y]$$

If A is integrable, $N_A = 0$.

The Nijenhuis Tensor

Need it for bi-Lagrangian structure $K = \eta^{-1}\omega$

$$
\begin{aligned}
N_K(X,Y,Z) &= \mathring{\nabla}_Y \omega(X, K(Z)) - \mathring{\nabla}_X \omega(Y, K(Z)) \\
&\qquad + \mathring{\nabla}_{K(Y)} \omega(X, Z) - \mathring{\nabla}_{K(X)} \omega(Y, Z) \\
&= \mathrm{d}\omega(K(X), K(Y), K(Z)) + \mathrm{d}\omega(X, Y, K(Z)) \\
&\qquad + 2\mathring{\nabla}_{K(Z)} \omega(X, Y)
\end{aligned}
$$

A Connection for Born Geometry

Connections for various geometries

Levi-Civita Connection

- Riemannian geometry: metric η
- Unique, compatible, symmetric, torsion-free

Fedosov Connection

- Symplectic geometry: almost symplectic form ω
- Family of compatible, symmetric, torsion-free

Bismut Connection

- Hermitian geometry: almost complex structure I, metric $\mathcal{H}$
- Unique, compatible with $\mathcal{H}$ & I, totally skew torsion

Connections for various geometries

Fedosov + metric = Levi-Civita

- Symplectic manifold with metric (real structure $K = \eta^{-1}\omega$)
- Unique, compatible with ω & η, symmetric, torsion-free

Doubled space of DFT

- Two metrics: η and $\mathcal{H}$ (chiral structure $J = \eta^{-1}\mathcal{H}$)
- DFT connection not fully determined

Born Connection

- Born geometry $(\mathcal{P}; \eta, \omega, \mathcal{H})$ with para-quaternions (I, J, K)
- Unique, compatible, (generalized) torsion is chiral

The Born Connection

Defining Properties

- Compatibility with Born geometry $(\eta, \omega, \mathcal{H})$

$$\nabla\eta = \nabla\omega = \nabla\mathcal{H} = 0$$

- Generalized torsion is chiral

$$\mathcal{T} \sim \mathcal{J}^{+} N_K$$

Vanishing generalized torsion $\mathcal{T}$ if K integrable

$$N_K = 0 \quad \Rightarrow \quad \mathcal{T} = 0$$

The Born Connection

Born Connection ∇ given by

$$\eta(\nabla_X Y, Z) = \eta(\mathring{\nabla}_X Y, Z) + \eta(X, \Omega(Y, Z))$$

where

- $\mathring{\nabla}$ is the Levi-Civita connection of η
- Ω is the contorsion

The Born Connection

Contorsion

$$\begin{aligned}\eta(X,\Omega(Y,Z)) = &\frac{1}{2}\mathring{\nabla}_X H(Y,J(Z))\\ &+\frac{1}{4}[\mathring{\nabla}_Y\mathcal{H}(J(Z),X) - \mathring{\nabla}_{K(Y)}\mathcal{H}(I(Z),X)\\ &\quad - \mathring{\nabla}_{J(Z)}\mathcal{H}(X,Y) + \mathring{\nabla}_{I(Z)}\mathcal{H}(X,K(Y))]\\ &-\frac{1}{4}[\mathring{\nabla}_Z\mathcal{H}(X,J(Y)) - \mathring{\nabla}_{K(Z)}\mathcal{H}(X,I(Y))\\ &\quad - \mathring{\nabla}_{J(Y)}\mathcal{H}(Z,X) + \mathring{\nabla}_{I(Y)}\mathcal{H}(K(Z),X)]\\ &+\frac{1}{4}[\mathring{\nabla}_{J(X)}\omega(I(Y),Z) + \mathring{\nabla}_{J(X)}\omega(Y,I(Z))\\ &\quad - \mathring{\nabla}_X\omega(Y,K(Z)) + \mathring{\nabla}_X\omega(J(Y),I(Z))]\end{aligned}$$

Properties and Identities

e.g. η-compatibility $\Rightarrow$ skew-symmetry

$$\Omega(X,Y) = -\Omega(Y,X)$$

Some identites needed for proofs

$$\begin{aligned} \mathring{\nabla}_X \mathcal{H}(Y,Z) &= \Omega(X,Y,J(Z))) - \Omega(X,J(Y),Z), \\ -\mathring{\nabla}_X \omega(Y,Z) &= \Omega(X,Y,K(Z)) + \Omega(X,K(Y),Z) \end{aligned}$$

Existence and Uniqueness

Uniqueness

- If such a connection exists, it is unique and given by Ω
- Fully determined in terms of $(\eta, \omega, \mathcal{H})$

Existence

- Constructive proof
- Properties of a connection

Vanishing Generalized Torsion

Need the following objects

- Chiral Nijenhuis tensor for K

$$\mathcal{J}^+ N_K(X,Y,Z)$$

measures integrability along the chiral subspaces $C_\pm$

- Polarized component of generalized torsion

$$\mathcal{K}^+ \mathcal{T}(X,Y,Z) = \frac{1}{2} \sum_{\mathrm{cycl}(X,Y,Z)} N_K(X,Y,Z)$$

Vanishing Generalized Torsion

Can express $\mathcal{T}$ in terms of N_K

$$\begin{aligned}\mathcal{T}(X,Y,Z) &= \mathcal{J}^{+}\mathcal{K}^{+}\mathcal{T}(X,Y,Z)\\ &= \frac{1}{2}\sum_{\mathrm{cycl}(X,Y,Z)} \mathcal{J}^{+}N_K(X,Y,Z)\end{aligned}$$

Generalized torsion vanishes if K is integrable

$$N_K = 0 \quad \Rightarrow \quad \mathcal{T} = 0$$

Fluxes

No flux: $B = 0$, $\mathrm{d}\omega = 0$

- $\mathring{\nabla}_X \omega(Y, Z) = 0$
- Levi-Civita = Fedosov connection

Turn on H-flux $H = \mathrm{d}B$

- $\mathcal{H} = \mathcal{H}(g, B)$ and $\mathrm{d}\omega \sim H \neq 0$
- flux appears in torsion ($\mathcal{T} \sim \mathcal{J}^+ N_K$ with $N_K \sim H$)
- Bismut connection - torsion is totally skew [Ellwood; Gualtieri]

More general fluxes $(H, f, Q, R) \subset \mathcal{F}$

- $\mathrm{d}\omega \sim \mathcal{F} \neq 0$

Structure group

Have the following groups

$$O(d,d) \cap Sp(2d) \cap O(2d,0) = O(d)$$

- Born connection reduces to $O(d)$ connection on Lagrangian submanifold with metric g
- Levi-Civita connection of g ?

Application to DFT

Coordinate Expression

Introduce frame field

- Frame field E_A
- Local coordinates $X = X^A E_A$

$$\eta_{AB} = \eta(E_A, E_B), \quad \omega_{AB} = \omega(E_A, E_B), \quad \mathcal{H}_{AB} = \mathcal{H}(E_A, E_B)$$

Covariant derivative of a vector

$$\nabla_A X^B = \mathring{\nabla}_A X^B + \Omega_{AC}{}^B X^C$$

Coordinate Expression

Born connection given by

$$\Omega_{ABC} = \frac{1}{2}\mathring{\nabla}_A \mathcal{H}_{BD} J^D{}_C + \left(\delta^{[D}{}_{[B} J^{E]}{}_{C]} - K^{[D}{}_{[B} I^{E]}{}_{C]}\right) \mathring{\nabla}_D \mathcal{H}_{EA}$$
$$- \frac{1}{2}\mathring{\nabla}_D \omega_{E[B} I^E{}_{C]} J^D{}_A - \frac{1}{4}\left(\delta^D{}_B K^E{}_C - J^D{}_B I^E{}_C\right) \mathring{\nabla}_A \omega_{DE}$$

Application to Double Field Theory

DFT Limit of Born Geometry: η and ω flat

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$$

Connection reduces to ($\mathring{\nabla} \to \partial$)

$$\Omega_{ABC} = \frac{1}{2}\partial_A \mathcal{H}_{BD} J^D{}_C + \frac{1}{2}\left(\delta^D{}_{[B} J^E{}_{C]} + J^D{}_{[B}\delta^E{}_{C]}\right)\partial_D \mathcal{H}_{EA} - \frac{1}{2}\left(K^D{}_{[B} I^E{}_{C]} - K^E{}_{[B} I^D{}_{C]}\right)\partial_D \mathcal{H}_{EA}$$

Application to Double Field Theory

Determined part of DFT connection [Coimbra, Strickland-Constable, Waldram; Hohm, Zwiebach; Jeon, Lee, Park]

$$\Gamma_{MNK} = \frac{1}{2}\partial_M \mathcal{H}_{NL} J^L{}_K + \frac{1}{2}\left(\delta^P{}_{[N} J^Q{}_{K]} + J^P{}_{[N}\delta^Q{}_{K]}\right)\partial_P \mathcal{H}_{QM}$$
$$+ \frac{2}{D-1}\left(\eta_{M[N}\delta^L{}_{K]} + \mathcal{H}_{M[N} J^L{}_{K]}\right)\left(\partial_L d + \frac{1}{4}\mathcal{H}^{PQ}\partial_Q \mathcal{H}_{PL}\right)$$
$$+ \hat{\Gamma}_{MNK}$$

DFT dilaton d not yet included

$$O(d,d) \to O(d,d) \times \mathbb{R}^+$$

Summary

- Born geometry $(\mathcal{P};\eta,\omega,\mathcal{H}) \to$ dynamical phase space
- Unique, compatible connection $\nabla = \mathring{\nabla} + \Omega$
  with chiral torsion $\mathcal{T} = \mathcal{J}^{+} N_K$
- Integrability condition: $N_K = 0 \quad \Rightarrow \quad \mathcal{T} = 0$
- DFT limit of Born geometry reproduces DFT connection

Doubled String Model

Setting the Scale

- length scale: $\lambda = \sqrt{\hbar \alpha'}$
- energy scale: $\epsilon = \sqrt{\hbar / \alpha'}$

$$\alpha' = \lambda / \epsilon$$

Doubled Coordinates

$$X^A = \begin{pmatrix} x^\mu / \sqrt{\alpha'} \\ y_\mu \sqrt{\alpha'} \end{pmatrix} = \begin{pmatrix} x^\mu / \lambda \\ y_\mu / \epsilon \end{pmatrix} = \begin{pmatrix} \frac{2\pi}{R} x^\mu \\ R \tilde{x}_\mu \end{pmatrix}$$

Quasi-periodicity and Monodromies

String action needs to be single-valued

- $\mathrm{d}X^\mu(\sigma, \tau)$ is periodic
- not necessary that $X^\mu(\sigma, \tau)$ is periodic

Quasi-periodic

$$X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau) + \tilde{p}^\mu$$

- if $\tilde{p} \neq 0 \rightarrow$ no a priori geometrical interpretation of closed string propagating in flat spacetime
- for compact, spacelike direction $\rightarrow \tilde{p}$ is interpreted as winding

Quasi-periodicity and Monodromies

In the dual picture also have

- $$\oint \star \mathrm{d}Y = \tilde{p}/\alpha' = \oint \mathrm{d}X$$
- $$\oint \star \mathrm{d}X = \alpha' p = \oint \mathrm{d}Y$$