Dirac geometry and the integration of Poisson homogeneous spaces

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(Joint work with H. Bursztyn and J.H. Lu)

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Symplectic groupoids

When does a Poisson manifold (M, π) arise from
a symplectic groupoid $(\mathcal{G}, \omega) \rightrightarrows M$

Characterization of integrability: Crainic-Fernandes
$A \cong T^*M$

Description of symplectic groupoids:

- Poisson Lie groups and affine Poisson structures (Lu-Weinstein)
- Integrability of quotients M/G (Fernandes-Ortega-Ratiu)

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Definition

Let H be a closed subgroup of G. A (G, π_G)**-homogeneous Poisson structure** on G/H is a Poisson bivector field π on G/H, and the action is Poisson

$$\sigma\colon \quad (G, \pi_G) \times (G/H, \pi) \longrightarrow (G/H, \pi), \quad (g_1, g_2H) \longmapsto g_1 g_2 H$$

Are Poisson homogeneous spaces integrable?
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Dirac structures. Presymplectic groupoids

$\mathbb{T}M := TM \oplus T^*M$

- nondegenerate fibrewise bilinear form given at each $x \in M$

$$\langle (X,\alpha),(Y,\beta)\rangle = \beta(X) + \alpha(Y), \qquad \alpha,\beta \in T_x^*M, X, Y \in T_xM.$$

- Courant bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(\mathbb{T}M)$

$$[\![(X,\alpha),(Y,\beta)]\!] = ([X,Y], \mathcal{L}_X\beta - i_Y d\alpha).$$

We denote by $\mathrm{pr}_T : \mathbb{T}M \to TM$ and $\mathrm{pr}_{T^*} : \mathbb{T}M \to T^*M$ the canonical projections.

Definition

A **Dirac structure** on M is a vector subbundle $E \subset \mathbb{T}M$ which is maximally isotropic with respect to $\langle\cdot,\cdot\rangle$ (that is, $E = E^{\perp}$) and which is involutive with respect to $[\![\cdot,\cdot]\!]$.

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$$E \text{ Dirac structure} \Rightarrow \begin{cases} (E, [\cdot,\cdot], pr_{T|E}) \text{Lie algebroid} \\ \varphi := pr_{T^*|E} : E \to T^*M \text{ closed IM 2-form} \end{cases}$$

$$\mathrm{Ker}(E) := E \cap TM \subseteq TM.$$

Example

(M, π) Poisson $\Rightarrow (M, E_\pi = \{(\pi^\sharp(\alpha), \alpha) \mid \alpha \in T^*M\})$ Dirac

$$\mathrm{Ker}(E) = \{0\} \Leftrightarrow E = E_\pi$$

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Given two Dirac manifolds (M_1, E_1) and (M_2, E_2) and a map $J\colon M_1 \to M_2$

i) J is a **forward Dirac map** if $E_2 = (J_* E_1)$, where

$$(J_* E_1) = \{(dJ(X), \beta) \mid (X, (dJ)^* \beta) \in E_1\}.$$

J is a **strong Dirac map** if $\mathrm{Ker}(dJ) \cap \mathrm{Ker}(E_1) = \{0\}$.

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A closed 2-form $\omega \in \Omega^2(\mathcal{G})$ is **multiplicative** if

$$m^*\omega = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega,$$

where $\mathrm{pr}_i : \mathcal{G}^{(2)} \to \mathcal{G}$, $i = 1, 2$, are the natural projections.

$$\left\{\text{closed multiplicative 2-forms on } \mathcal{G}\right\} \Leftrightarrow \left\{\text{closed IM 2-forms on } A\right\}$$

$$\varphi(a)(X) = \omega(a, X), \qquad a \in A, X \in TM,$$

If (M, E) is a Dirac manifold (E integrable Lie algebroid) then

$$\mathrm{Ker}(\omega_m) \cap \mathrm{Ker}(ds)_m \cap \mathrm{Ker}(dt)_m = \{0\}, \qquad m \in M$$

$(\mathcal{G}, \omega)$ is called a **presymplectic groupoid**.

$$\mathrm{Ker}(\omega) = \{a^r - \mathrm{inv}(b)^l \mid a, b \in \mathrm{Ker}(E)\}.$$

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Invariant Dirac structures and Poisson quotients

$H \circlearrowleft M$ free and proper
$q : M \to M/H$ submersion

$$\rho_M : \mathfrak{h} \times M \to TM$$

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Proposition

H acts by Dirac maps and the distribution tangent to the H-orbits agrees with Ker(E):

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Proposition

(a) The operations of pullback and pushforward establish a one-to-one correspondence between Poisson structures π on M/H and H-invariant Dirac structures E on M satisfying $\mathrm{Ker}(E) = \rho_M(\mathfrak{h} \times M)$.

(b) Let M_1 and M_2 be manifolds, carrying free and proper H-actions. For $i = 1, 2$, let π_i be a Poisson structure on M_i/H with corresponding Dirac structure E_i on M_i. Consider an H-equivariant map $f : M_1 \to M_2$ covering $\bar{f} : M_1/H \to M_2/H$. Then $\bar{f}$ is a Poisson map if and only if f is a strong Dirac map.

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Integrability of Poisson structures on quotients

$(\overline{\mathcal{G}}, \bar{\omega})$ symplectic groupoid integrating $(M/H, \pi)$ with source and target maps $\bar{\mathbf{s}}, \bar{\mathbf{t}} : \overline{\mathcal{G}} \to M/H$.

$(q^!\overline{\mathcal{G}}, p^*\bar{\omega})$ presymplectic groupoid integrating the Dirac structure E on M

$$q^!\overline{\mathcal{G}} = \{(x_1, \bar{g}, x_2) \in M \times \overline{\mathcal{G}} \times M \mid \bar{\mathbf{s}}(\bar{g}) = q(x_2),\ \bar{\mathbf{t}}(\bar{g}) = q(x_1)\}$$

Example

$M = S^3 \times \mathbb{R}$, $S^1 \circlearrowright M$

$$q : M \to M/S^1 = S^2 \times \mathbb{R}$$

$(S^2 \times \mathbb{R}, \pi)$ Poisson s.t. $(S^2 \times \{t\}, (1+t^2)\omega_{S^2})$ symplectic leaves
However, (M, E) is integrable (regular & $\pi_2(S^3)$ vanishes).

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$$\psi\colon \mathfrak{h}\times M \to E, \qquad (u,x)\mapsto (u_M(x),0),$$

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$(\mathcal{G},\omega)$ a presymplectic groupoid integrating E

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$$\rho_{\mathcal{G}}\colon \mathfrak{h}\times\mathfrak{h}\to \mathfrak{X}(\mathcal{G}), \qquad (u,v)\mapsto (\psi(u))^r + (\mathrm{inv}(\psi(v)))^l$$

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If $\exists \Psi : H\ltimes M \to \mathcal{G}$ then there exists a $(H\times H)$-action on $\mathcal{G}$

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Proposition

Let π be a Poisson structure on M/H, and let E be its pullback on M. Let $\mathcal{G}$ be a Lie groupoid integrating E as a Lie algebroid for which the morphism ψ in integrates to $\Psi\colon H \ltimes M \to \mathcal{G}$, and consider the previous $(H \times H)$-action on $\mathcal{G}$.

(a) The orbit space $\mathcal{G}/(H \times H)$ is a Lie groupoid (such that the quotient projection is a groupoid morphism) integrating the Lie algebroid $T^*(M/H)$ defined by π.

(b) If H is connected and if ω is a multiplicative 2-form on $\mathcal{G}$ such that $(\mathcal{G}, \omega)$ is a presymplectic integration of E as a Dirac structure, then there is a unique symplectic structure $\bar{\omega}$ on $\mathcal{G}/(H \times H)$ with $p^*\bar{\omega} = \omega$ and $(\mathcal{G}/(H \times H), \bar{\omega})$ is a symplectic groupoid over $(M/H, \pi)$.

Note

If $(M/H, \pi)$ is integrable with symplectic groupoid $\overline{\mathcal{G}}$ then $\mathcal{G} = q^! \overline{\mathcal{G}}$ is a presymplectic groupoid over M. In addition, the morphism ψ integrates to the Lie groupoid morphism

$$\Psi \colon H \ltimes M \to \mathcal{G}, \;\; \Psi(h, x) = (hx, \bar{\iota}_{q(x)}, x),$$

and the induced $H \times H$ action on $\mathcal{G}$ is

$$((h_1, h_2), (x_1, \bar{g}, x_2)) \mapsto (h_1 x_1, \bar{g}, h_2 x_2).$$

In particular, $\mathcal{G}/(H \times H) = \overline{\mathcal{G}}$.

If $\widetilde{\mathcal{G}}$ is the source-simply connected presymplectic groupoid integrating E and H is connected

$$\widetilde{\Psi} : \widetilde{H} \ltimes M \to \widetilde{\mathcal{G}},$$

where $\widetilde{H}$ is the simply-connected Lie group integrating $\mathfrak{h}$.
If $\widetilde{\Psi}(\pi_1(H) \times M) \subset \mathcal{G}$ is an embedded submanifold, then it is an normal Lie subgrupoid and

$$\mathcal{G} := \widetilde{\mathcal{G}}/\widetilde{\Psi}(\pi_1(H) \times M)$$

is a Lie groupoid over M. Moreover, since this image is isotropic with respect to the presymplectic form ω on $\widetilde{\mathcal{G}}$ implies that ω descends to $\mathcal{G}$, making it into a presymplectic groupoid integrating E and $\widetilde{\Psi}$ descends to a morphism $\Psi : H \ltimes M \to \mathcal{G}$ integrating ψ.

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Proposition

Let H be a Lie group, and H_0 be its connected component containing the identity. Let M be a manifold carrying an H-action that is free and proper. Let π be a Poisson structure on M/H and E the Dirac structure on M obtained by pullback. Suppose that E is integrable, that $\widetilde{\mathcal{G}}$ is the source-simply connected presymplectic groupoid integrating E and that the image of the map $\pi_1(H_0) \times M \to \widetilde{\mathcal{G}}$ is an embedded submanifold. Then π is integrable.

Integration of Poisson homogeneous spaces

(G, π_G) Poisson Lie group

$$m : (G, \pi_G) \times (G, \pi_G) \to (G, \pi_G)$$

Poisson map.
$\delta : \mathfrak{g} \to \wedge^2\mathfrak{g}$ is the linearization of π_G at e (i.e., $\delta(u) = (\mathcal{L}_{u^r}\pi_G)_e$)
then the dual map

$$\delta^* : \ \wedge^2 \mathfrak{g}^* \longrightarrow \mathfrak{g}^*, \ \ \xi \wedge \eta \longmapsto [\xi, \eta]_{\mathfrak{g}^*},$$

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$\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ Lie algebra

$[u + \xi,\, v + \eta]_{\mathfrak{d}} = [u, v]_{\mathfrak{g}} + \mathrm{ad}^*_{\xi} v - \mathrm{ad}^*_{\eta} u + [\xi, \eta]_{\mathfrak{g}^*} + \mathrm{ad}^*_{u} \eta - \mathrm{ad}^*_{v} \xi,$

for $u, v \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$, and the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{d}$ given by

$$\langle u + \xi,\ v + \eta \rangle = \eta(u) + \xi(v),$$

is ad-invariant with respect to $[\cdot, \cdot]_{\mathfrak{d}}$
$(\mathfrak{d}, \langle \cdot, \cdot \rangle)$ is the **double** of the Lie bialgebra $(\mathfrak{g}, \delta)$

The adjoint action of $\mathfrak{g}$ on $\mathfrak{d}$ integrates to an action of G on $\mathfrak{d}$, still denoted by $\mathrm{Ad}_g : \mathfrak{d} \to \mathfrak{d}$ for $g \in G$, which is given by

$$\mathrm{Ad}_g(u + \xi) = \mathrm{Ad}_g u + i_{\mathrm{Ad}^*_{g^{-1}} \xi}((r_{g^{-1}})_* \pi_G|_g) + \mathrm{Ad}^*_{g^{-1}} \xi,$$

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$H \circlearrowleft G \qquad (h, g) \mapsto r_{h^{-1}}(g) = gh^{-1}$

$$\rho : \mathfrak{h} \times G \to TG, \quad \rho(u, g) = -u^l|_g = -dl_g|_e(u).$$

Proposition

Homogeneous Poisson structures on G/H are in one-to-one correspondence, via pullback/pushforward by $q : G \to G/H$, with Dirac structures E on G satisfying:

(i) E is H-invariant,

(ii) $\mathrm{Ker}(E) = \rho(\mathfrak{h} \times G)$,

(iii) $m : (G, \pi_G) \times (G, E) \to (G, E)$ is a (forward) Dirac map.

Let H be a closed subgroup of G. A (G, π_G)**-homogeneous Poisson structure** on G/H is a Poisson bivector field π on G/H, and the action is Poisson

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Dressing action of $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ on G,

$$\rho_{\mathfrak{d}} : \mathfrak{d} \to \mathfrak{X}(G), \quad u + \xi \mapsto -(u^l + \pi_G^\sharp(\xi^l)),$$

$\mathfrak{d}_G = \mathfrak{d} \times G \to G$ Courant algebroid

$$\rho_{\mathfrak{d}} \circ \mathrm{Ad}_g = dr_{g^{-1}} \circ \rho_{\mathfrak{d}}, \qquad g \in G.$$

$$I : \mathfrak{d}_G \xrightarrow{\cong} TG \oplus T^*G, \quad I(u + \xi) = -(u^l + \pi_G^\sharp(\xi^l), \xi^l).$$

$G \circlearrowleft G$, $g \mapsto (r_{g^{-1}} : G \to G)$, consider its natural lift to $\mathbb{T}G$
$g \mapsto (dr_{g^{-1}}, (r_g)^*)$.

Lemma

(a) The map $I : \mathfrak{d}_G \to \mathbb{T}G$ is G-equivariant,

(b) Every lagrangian subalgebra $\mathfrak{l} \subset \mathfrak{d}$ defines a Dirac structure $\mathfrak{l}_G := \mathfrak{l} \times G$ in the Courant algebroid $\mathfrak{d}_G$ whose underlying Lie algebroid is the action Lie algebroid defined by the dressing action restricted to $\mathfrak{l}$.

Lemma

A Dirac structure E on G is of the form $I(\mathfrak{l}_G)$ for a lagrangian subalgebra $\mathfrak{l} \subset \mathfrak{d}$ if and only if the group multiplication $m : (G, \pi_G) \times (G, E) \to (G, E)$ is a (forward) Dirac map. In this case, $\mathfrak{l} = E|_e$.

In conclusion, we have

Proposition

The map $I : \mathfrak{d}_G \to \mathbb{T}G$ establishes a one-to-one correspondence between Dirac structures on G satisfying properties (i), (ii) and (iii) and lagrangian subalgebras $\mathfrak{l} \subset \mathfrak{d}$ which are Ad_H-invariant and satisfy $\mathfrak{l} \cap \mathfrak{g} = \mathfrak{h}$.

$$\mathfrak{l} = \{u + \xi \,|\, u \in \mathfrak{g},\ \xi \in \mathrm{Ann}(\mathfrak{h}), i_\xi(\pi|_{q(e)}) = u + \mathfrak{h}\}$$

G/H Poisson homogeneous space.

i) $q^*(E_\pi) \cong \mathfrak{l} \ltimes G$ integrable Lie algebroid

ii) $\psi \colon \mathfrak{h} \ltimes G \to \mathfrak{l} \ltimes G$, $(u, g) \mapsto (u + 0, g)$, $u \in \mathfrak{h}$
$\widetilde{\Psi}(\pi_1(H_0) \times G) \subseteq \widetilde{\mathcal{G}}$ embedded submanifold.

Theorem

Any Poisson homogeneous space $(G/H, \pi)$ is integrable.

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Any Poisson homogeneous space $(G/H, \pi)$ is integrable.

Examples

The complete case: $\rho_{\mathfrak{d}}|_{\mathfrak{l}} \colon \mathfrak{l} \to \mathfrak{X}(G)$ is by complete vector fields

$\widetilde{\mathcal{G}} := L \ltimes G$ action Lie groupoid

If $H \subseteq L$ is a closed Lie subgroup then

$$\Psi \colon H \times G \to L \times G, \qquad (h, g) \mapsto (h, g).$$

$$(H \times H) \times (L \times G) \to L \times G, \ (h_1, h_2) \cdot (l, g) = (h_1 l h_2^{-1}, g h_2^{-1}).$$

$\widetilde{\mathcal{G}}/(H \times H) \cong G \times_H (L/H)$

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$\widetilde{\mathcal{G}}/(H \times H) \cong G \times_H (L/H)$

D be a connected Lie group integrating $\mathfrak{d}$
G^* integrating $\mathfrak{g}^*$ is closed in D, $G^* \to D, v \mapsto \bar{v}$
$G \to D, g \mapsto \bar{g}$
L connected Lie group with Lie algebra $\mathfrak{l}$, $L \to D, l \mapsto \bar{l}$
H also embeds into L as a closed subgroup such that every $h \in H$ has the same image under $L \to D$ and $G \to D$.

$$\mathcal{G}(L) = \{(v, g_1, g_2, l) \in G^* \times G \times G \times L \,|\, \bar{v}\bar{g}_1 = \bar{g}_2\bar{l}^{-1}\}.$$

$\mathcal{G}(L)$ is a Lie groupoid over G with structure maps

$$\mathbf{s}(v, g_1, g_2, l) = g_2, \quad \mathbf{t}(v, g_1, g_2, l) = g_1,$$

and multiplication $\mathcal{G}(L)^{(2)} \to \mathcal{G}(L)$,

$$(v_1, g_1, g, l_1) \cdot (v_2, g, g_2, l_2) = (v_2 v_1, g_1, g_2, l_1 l_2).$$

Proposition

1) The Lie groupoid $\mathcal{G}(L)$ is an integration of the Lie algebroid $\mathfrak{l} \ltimes G$;
2) With $H \times H$ act on $\mathcal{G}(L)$ by

$$(h_1, h_2) \cdot (v, g_1, g_2, l) = (v, g_1 h_1^{-1}, g_2 h_2^{-1}, h_1 l h_2^{-1}),$$

the quotient $\mathcal{G}(L)/(H \times H)$ is a Lie groupoid over G/H integrating the Lie algebroid $T^*(G/H)$ defined by the Poisson structure on G/H.