The Structure of Mapping Objects in the Category of Orbifolds

Dorette Pronk\textsuperscript{1} with Laura Scull\textsuperscript{2}

\textsuperscript{1}Dalhousie University, Halifax, NS
\textsuperscript{2}Fort Lewis College, Durango, CO

Geometric Structures on Lie Groupoids
BIRS

April 18, 2017
Main references for this talk:


- Dorette Pronk, Laura Scull, A Bicategory of Orbigroupoids with Small Hom-Groupoids, in progress.

- Dorette Pronk, Laura Scull, Exponential Objects for Orbigroupoids, in progress.
Given two orbifolds $\mathcal{G}$ and $\mathcal{H}$, can we put a topology on the hom-groupoid $\text{OMap}(\mathcal{G}, \mathcal{H})$ of good maps and 2-cells?

When does this hom-groupoid represent a (possibly infinite dimensional) orbifold?

When does it have the universal property to be a categorical exponential?
Outline

1. Orbigroupoids
2. Maps Between Orbigroupoids
3. Orbispaces
4. Small mapping groupoids
5. The Topology on Mapping Groupoids
6. Mapping Objects with Compact Domain
The smooth case
A smooth orbigroupoid is a Lie groupoid with
- structure maps that are local diffeomorphisms;
- a proper diagonal $G_1 \rightarrow G_0 \times G_0$.

The topological case
An orbigroupoid is a topological groupoid with
- structure maps that are étale;
- a proper diagonal $G_1 \rightarrow G_0 \times G_0$. 
Orbigroupoids

Remarks

1. The isotropy groups of an orbigroupoid are finite.
2. The quotient space, 

\[ \mathcal{G}_1 \xrightarrow{s} \mathcal{G}_0 \xrightarrow{t} \mathcal{G}_0/\mathcal{G}_1 \]

is also called the **underlying space** of the orbigroupoid.

3. Properness of the groupoid implies that this quotient space is Hausdorff.

4. For this talk I work with the topological case.

5. And I take **Top** to be a Cartesian closed category of topological spaces.
Examples: a $G$-point $*_G$
Examples
A Cone of Order 3

This is a translation groupoid, $\mathbb{Z}/3 \ltimes D$. 
Examples
The Unit Interval

morphisms

objects
Examples
A Split Unit Interval

\[ \begin{array}{c}
\text{morphisms} \\
\text{objects}
\end{array} \]
Examples
The Teardrop Groupoid
Examples: The Triangular Billiard Groupoid

**Objects:**
- Quotient
- $\sigma$
- $\sigma \rho$
- $\sigma \rho^2$
(glueing via natural transformation)

**Billiard groupoid**

**Morphisms:**
- $D_3 \times 36 \times$
- $D_3 \times 36 \times$
- $D_3 \times 36 \times$

**Objects:**
- $\mathbb{Z}/2$
Examples: The $\mathbb{Z}/3$ Circles, $S^1_{\mathbb{Z}/3}$ and $\tilde{S}^1_{\mathbb{Z}/3}$
Good Maps Between Orbigroupoids

There are two approaches to obtain a bicategory of orbigroupoids that is appropriate for homotopy theory.

- Hilsum-Skandalis bibundles with bundle isomorphisms
- The bicategory of fractions of continuous groupoid homomorphisms with respect to essential equivalences
- The two approaches give biequivalent bicategories of orbigroupoids.
Maps Between Orbigroupoids

Continuous Groupoid Homomorphisms

\[ \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \longrightarrow \mathcal{G}_1 \quad \text{and} \quad \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \]

\[ m \quad \text{and} \quad m \]

\[ f_1 \times f_1 \quad \text{and} \quad f_1 \]

\[ G_1 \quad \text{and} \quad H_1 \]

\[ f_0 \quad \text{and} \quad f_0 \]

\[ G_0 \quad \text{and} \quad H_0 \]
2-Cells

A 2-cell

$$\alpha : f \Rightarrow f' : \mathcal{G} \Rightarrow \mathcal{H}$$

is given by, a continuous function

$$\alpha : \mathcal{G}_0 \to \mathcal{H}_1$$

such that

- $s \circ \alpha = f_0$ and $t \circ \alpha = f'_0$;
- (naturality) the following square commutes in $\mathcal{H}$ for each $g \in \mathcal{G}_1$,

$$
\begin{array}{ccc}
  f_0(sg) & \xrightarrow{f_1(g)} & f_0(tg) \\
  \alpha(sg) \downarrow & & \downarrow \alpha(tg) \\
  f'_0(sg) & \xrightarrow{f'_1(tg)} & f'_0(tg)
\end{array}
$$
Let $\mathcal{G}$ and $\mathcal{H}$ be topological groupoids.

The continuous groupoid homomorphisms from $\mathcal{G}$ to $\mathcal{H}$ and continuous natural transformations between them form a topological groupoid $\text{GMap}(\mathcal{G}, \mathcal{H})$. 

Example: $\text{GMap}(*_{\mathbb{Z}/2}, T)$

\[ \begin{array}{c}
\begin{array}{c}
\ast \\
\tau
\end{array} \quad \xrightarrow{f} \quad \begin{array}{c}
\ast \\
\ast A
\end{array} \\
\ast \\
\ast I
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\ast \\
\ast A
\end{array} \\
\ast \\
\ast I
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\ast \\
\ast A
\end{array} \\
\ast \\
\ast I
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\ast \\
\ast A
\end{array} \\
\ast \\
\ast I
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\ast \\
\ast A
\end{array} \\
\ast \\
\ast I
\end{array} \end{array} \]
Example: \( \text{GMap}(*\mathbb{Z}/2, T) \)

- We obtain a copy of the original orbigroupoid \( T \) together with a copy of the (trivial) \( \mathbb{Z}/2 \)-circle, \( S^1_{\mathbb{Z}/2} \).
From Orbigroupoids to Orbispaces

- The following two groupoids both represent the unit interval as orbispace

```
  objects          objects
  morphisms       morphisms
```

- They are not isomorphic in the category of orbigroupoids and groupoid homomorphisms.
- However, the groupoid homomorphism from the second to the first is an essential equivalence.
Essential Equivalences

- A morphism $\varphi: \mathcal{G} \to \mathcal{H}$ is an **essential equivalence** when it is essentially surjective and fully faithful.

- It is **essentially surjective** when $\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \rightarrow \mathcal{H}_0$ in

  \[
  \begin{array}{ccc}
  \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 & \longrightarrow & \mathcal{H}_1 \\
  \downarrow & & \downarrow t \\
  \mathcal{G}_0 & \varphi_0 & \mathcal{H}_0
  \end{array}
  \]

  is an **open surjection**.

\[
\begin{array}{cc}
\mathbf{G}_{\text{obj}} & \longrightarrow \\
\downarrow & \downarrow \\
\mathbf{H}_{\text{obj}} & \\
\end{array}
\]
The morphism \( \varphi: \mathcal{G} \to \mathcal{H} \) is **fully faithful** when

\[
\begin{array}{c}
G_1 \xrightarrow{\varphi_1} H_1 \\
\downarrow (s,t) \quad \quad \downarrow (s,t) \\
G_0 \times G_0 \xrightarrow{\varphi_0 \times \varphi_0} H_0 \times H_0
\end{array}
\]

is a pullback,
Morita Equivalence

- The essential equivalence maps between topological groupoids generate the **Morita equivalence relation** in the sense that $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent if and only if they are connected by a span of essential equivalence maps,

  \[
  \mathcal{G} \xleftarrow{\varphi} \mathcal{K} \xrightarrow{\psi} \mathcal{H}
  \]

- To define a bicategory of orbispaces, we use a **bicategory of fractions** to invert the essential equivalences.
Maps are **generalized maps** defined by spans

\[ \mathcal{G} \leftrightarrow K \xrightarrow{\varphi} \mathcal{H} \]

where \( \nu \) is an essential equivalence.

A **2-cell** between two generalized maps is an (equivalence class of) diagrams

where \( \nu \nu_1 \) is an essential equivalence.
Example
Recall: we want to define a topological groupoid $\text{OMap}(\mathcal{G}, \mathcal{H})$ with

- objects given by generalized maps $\mathcal{G} \xleftarrow{\nu} \mathcal{K} \xrightarrow{\phi} \mathcal{H}$
- arrows given by 2-cells; i.e., equivalence classes of diagrams

The collection of generalized maps $\mathcal{G} \to \mathcal{H}$ as described is a proper class.

How do we get a good description of the space of arrows, $\text{OMap}(\mathcal{G}, \mathcal{H})_1$? This is a quotient of the space of 2-cell diagrams!
Recall: we want to define a **topological groupoid** $\text{OMap}(G, H)$ with

- objects given by generalized maps $G \leftarrow K \rightarrow H$
- arrows given by 2-cells; i.e., equivalence classes of diagrams

The collection of generalized maps $G \to H$ as described is a **proper class**.

How do we get a good description of the space of arrows, $\text{OMap}(G, H)_1$? This is a **quotient of the space of 2-cell diagrams**!
We introduce a smaller class of arrows to be inverted: essential covering maps.

This class does not satisfy the right bicalculus of fractions conditions (they are not closed under composition).

There is an adjusted version of these conditions and an adjusted bicategory of fractions construction to fix this.

However, you don’t need the full structure of the new bicategory of fractions to define the mapping groupoids in terms of essential covering maps and verify that they are equivalent, as categories, to the ones defined using essential equivalences.
We originally planned to obtain the topology on $\text{OMap}(G, H)$ as a pseudo-colimit of the groupoids $\text{GMap}(G', H)$, indexed over the diagram of essential equivalences over $G, G' \to G$.

This requires some work in the general case, and has been shown to work in some special cases (see Angel and Colman, Free and based path groupoids, arXiv)

While working on the pseudo-colimit, we looked closely at the bicategory of fractions.

Then we realized that the equivalence relation was not so unwieldy after all.

So we switched to using its properties for a more direct approach.
Essential Coverings

- A collection $\mathcal{U}$ of open subsets of $\mathcal{G}_0$ is an **essential covering** of $\mathcal{G}_0$ if the map $(j_\mathcal{U})_0 : \bigsqcup_{U \in \mathcal{U}} U \to \mathcal{G}_0$ is essentially surjective.

- Note that an essential covering does not necessarily cover all of $\mathcal{G}_0$, but it meets every orbit.
Essential Covering Maps

Any essential covering $\mathcal{U}$ gives rise to a groupoid $\mathcal{G}^*(\mathcal{U})$ with a groupoid homomorphism $j_{\mathcal{U}} : \mathcal{G}^*(\mathcal{U}) \to \mathcal{G}$:

- $\mathcal{G}^*(\mathcal{U})_0 = \bigsqcup_{U \in \mathcal{U}} U$;
- $(j_{\mathcal{U}})_0 : \mathcal{G}^*(\mathcal{U})_0 \to \mathcal{G}_0$ is defined by inclusions on the connected components;
- $\mathcal{G}^*(\mathcal{U})_1$ is defined as the pullback,

$$
\begin{array}{ccc}
\mathcal{G}(\mathcal{U})_1 & \xrightarrow{(j_{\mathcal{U}})_1} & \mathcal{G}_1 \\
\downarrow{(s,t)} & & \downarrow{(s,t)} \\
\bigsqcup_{U \in \mathcal{U}} U \times \bigsqcup_{U \in \mathcal{U}} U & \xrightarrow{(j_{\mathcal{U}})_0 \times (j_{\mathcal{U}})_0} & \mathcal{G}_0 \times \mathcal{G}_0.
\end{array}
$$

- This makes the map $j_{\mathcal{U}} : \mathcal{G}^*(\mathcal{U}) \to \mathcal{G}$ an essential equivalence.
Example
Definition

For an orbigroupoid $\mathcal{G}$, the collection of essential covering maps is obtained as follows:

- Take all essential coverings of $\mathcal{G}_0$ which form a subset of the powerset of $\mathcal{G}_0$; i.e., all open subsets in the cover are distinct. (The cover is non-repeating.)
- Take all maps $w : \mathcal{G}^*(U) \to \mathcal{G}$ for which there is a natural isomorphism $\alpha_w : w \cong j_U$. 
Properties of Essential Covering Maps

For any orbigroupoid $G$, there is a set of essential covering maps with codomain $G$. 
Properties of Essential Covering Maps

- For each essential equivalence $\mathcal{K} \xrightarrow{\nu} \mathcal{G}$ of orbigroupoids there is an essential covering $\mathcal{U}$ of $\mathcal{G}$ such that $j_\mathcal{U}$ factors through $\nu$,

$$
\begin{array}{ccc}
\mathcal{G}^*(\mathcal{U}) & \xrightarrow{j_\mathcal{U}} & \mathcal{G} \\
\downarrow{\varepsilon} & \searrow{j_\mathcal{U}} & \\
\mathcal{K} & \xrightarrow{\nu} & \mathcal{G}
\end{array}
$$

- Given two orbigroupoids $\mathcal{G}$ and $\mathcal{H}$, each generalized map

$$
\begin{array}{ccc}
\mathcal{G} & \xleftarrow{\nu} & \mathcal{K} \xrightarrow{\varphi} \mathcal{H}
\end{array}
$$

is isomorphic to one of the form,

$$
\begin{array}{ccc}
\mathcal{G} & \xleftarrow{j_\mathcal{U}=\nu\varepsilon} & \mathcal{G}^*(\mathcal{U}) \xrightarrow{\varphi'=\varphi\varepsilon} \mathcal{H}
\end{array}
$$
Properties of Essential Covering Maps

Any 2-cell from

\[
\begin{array}{ccc}
G & \xleftarrow{w} & G^*(U) \\
& \varphi \to & H
\end{array}
\]

to

\[
\begin{array}{ccc}
G & \xleftarrow{w'} & G^*(V) \\
& \psi \to & H
\end{array}
\]

can be represented by a diagram of the form

\[
\begin{array}{ccc}
G & \xleftarrow{w} & G^*(U) \\
& \xleftarrow{j^w_U} \uparrow & \varphi \to H \\
& \xleftarrow{j^w_V} \downarrow \beta & G^*(W) \\
& \xleftarrow{w'} \downarrow \psi & G^*(V) \\
& \xleftarrow{\alpha} & G
\end{array}
\]

The essential covering \( W \) can be viewed as an essential refinement of \( U \) and \( V \).
The Mapping Groupoid, $\text{OMap}(\mathcal{G}, \mathcal{H})$

Let $\text{OMap}(\mathcal{G}, \mathcal{H})$ be the groupoid such that each object corresponds to a span,

$$
\mathcal{G} \leftarrow \mathcal{G}^*(\mathcal{U}) \overset{f}{\rightarrow} \mathcal{H}
$$

and each arrow corresponds to an equivalence class of diagrams,

$$
\begin{array}{ccc}
\mathcal{G} & \overset{w}{\leftarrow} & \mathcal{G}^*(\mathcal{U}) \\
&w & \overset{\alpha}{\leftarrow} \\
&\mathcal{G} & \overset{\beta}{\rightarrow} & \mathcal{H} \\
&\mathcal{G}^*(\mathcal{V}) & \overset{\psi}{\rightarrow} & \mathcal{H} \\
&\mathcal{G}^*(\mathcal{W}) & \overset{\varphi}{\rightarrow} & \mathcal{H} \\
&\mathcal{G}^*(\mathcal{U}) & \overset{j_{\mathcal{U}}^w}{\leftarrow} & \mathcal{G}^*(\mathcal{U}) \\
\end{array}
$$

**Goal:** Give a relatively simple description of the topology on this groupoid.
The Space of Objects

- Write

\[ \text{CMap}(\mathcal{G}^*(U), \mathcal{G}) \subseteq \text{GMap}(\mathcal{G}^*(U), \mathcal{G}) \]

for the full subgroupoid of \( \text{GMap}(\mathcal{G}^*(U), \mathcal{G}) \) on the essential covering maps, with the subspace topology.

- Then we obtain

\[ \text{OMap}(\mathcal{G}, \mathcal{H})_0 = \bigsqcup_U \text{CMap}(\mathcal{G}^*(U), \mathcal{G})_0 \times \text{GMap}(\mathcal{G}^*(U), \mathcal{H})_0, \]

where the coproduct is taken over all non-repeating essential covers of \( \mathcal{G}_0 \).
The Equivalence Relation

Given any two generalized maps \((w, f), (w', f'): G \to H\) and ANY common essential refinement every 2-cell \((w, f) \Rightarrow (w', f')\) can be represented \textbf{uniquely} by a diagram of the form

\[\begin{array}{ccc}
G^*(U) & \xrightarrow{f} & H \\
\downarrow^{s_{U,U'}} & & \downarrow^{\beta} \\
\alpha_{w,w'} & & \beta \\
\downarrow_{t_{U,U'}} & & \\
G^*(U') & \xleftarrow{w'} & \xleftarrow{w}
\end{array}\]

for \textbf{this particular chosen} common refinement.
Main Result

- It is possible to choose the refinements
  
  \[
  \begin{align*}
  \mathcal{G} & \quad \mathcal{G}^*(U) \\
  \downarrow^{w} & \quad \downarrow^{s_{U,U'}} \\
  \mathcal{G} & \quad \mathcal{G}^*(V) \quad \mathcal{G}^*(U') \\
  \downarrow^{\alpha_{w,w'}} & \quad \downarrow^{t_{U,U'}} \\
  \end{align*}
  \]

  in such a way that the map from the space of all diagrams to the subspace of representatives with these refinements is continuous.

- Hence, we have a retract, and the quotient topology is the subspace topology.
Choice of Refinements

- Choose an essential common refinement

\[ G^*(W_{U,U'}) \xrightarrow{s_{U,U'}} G^*(U) \]

\[ t_{U,U'} \downarrow \quad \alpha_{U,U'} \downarrow \quad j_U \]

\[ G^*(U') \xrightarrow{j_{U'}} G \]

for each pair \( U, U' \) of essential coverings of \( G_0 \);

- For each essential covering map \( w: G^*(U) \to G \), choose a 2-cell \( \beta_w: w \Rightarrow j_U \).

- Choose the composites of the 2-cells \( \alpha_{U,U'} \) with the \( \beta_w \)'s to define the \( \alpha_{w,w'}: ws_{U,U'} \Rightarrow w't_{U,U'} \).
Composition with an Essential Equivalence

**Proposition**

If $\varphi : \mathcal{G}' \to \mathcal{G}$ is an essential equivalence, then the induced map

$\varphi^* : \text{GMap}(\mathcal{G}, \mathcal{H}) \to \text{GMap}(\mathcal{G}', \mathcal{H})$,

is fully faithful in the sense that

\[
\begin{array}{ccc}
\text{GMap}(\mathcal{G}, \mathcal{H})_1 & \longrightarrow & \text{GMap}(\mathcal{G}', \mathcal{H})_1 \\
\downarrow & & \downarrow \\
\text{GMap}(\mathcal{G}, \mathcal{H})_0 \times \text{GMap}(\mathcal{G}, \mathcal{H})_0 & \longrightarrow & \text{GMap}(\mathcal{G}', \mathcal{H})_0 \times \text{GMap}(\mathcal{G}', \mathcal{H})_0
\end{array}
\]

is a pullback of spaces.
The Topology on Mapping Groupoids

The Space of Arrows

Write $P_{u,u'}$ for the pseudo pullback of groupoids,

\[
\begin{array}{ccc}
P_{u,u'} & \longrightarrow & \text{GMap}(\mathcal{G}^*(U), H) \\
\downarrow & & \downarrow s_{u,u'}^* \quad \cong \\
\text{GMap}(\mathcal{G}^*(U'), H) & \longrightarrow & \text{GMap}(\mathcal{G}^*(W_{u,u'}), H).
\end{array}
\]

Then,

\[
\text{OMap}(\mathcal{G}, H)_1 \cong \bigsqcup_{u,u'} \text{CMap}(\mathcal{G}^*(U), \mathcal{G})_0 \times \text{CMap}(\mathcal{G}^*(U'), \mathcal{G})_0 \times (P_{u,u'})_0.
\]

In particular, this space is Hausdorff.
Composition

Proposition

Composition by a generalized map \((w, f) = G \xleftarrow{w} G^*(U) \xrightarrow{f} H\) induces continuous groupoid maps between mapping groupoids,

\[(w, f)^* : \text{OMap}(\mathcal{K}, G) \rightarrow \text{OMap}(\mathcal{K}, H)\]

and

\[(w, f)^* : \text{OMap}(\mathcal{H}, L) \rightarrow \text{OMap}(G, L).\]
Morita Invariance

Theorem

Whenever $\mathcal{G}$ and $\mathcal{G}'$ are Morita equivalent and $\mathcal{H}$ and $\mathcal{H}'$ are Morita equivalent, the corresponding mapping groupoids

$$\text{OMap}(\mathcal{G}, \mathcal{H}) \text{ and } \text{OMap}(\mathcal{G}', \mathcal{H}')$$

are (Morita) equivalent.
When $\mathcal{G}_0/\mathcal{G}_1$ is compact,

- We only need to consider finite essential coverings with compact closures.
- We obtain a groupoid $\text{OMap}_c(\mathcal{G}, \mathcal{H}) \hookrightarrow \text{OMap}(\mathcal{G}, \mathcal{H})$. 
The Space of Objects

- The space of essentially compact covering maps $G^*(U) \to G$ is discrete.
- Hence the space of objects has the form,

$$O\text{Map}_c(G, H)_0 \cong \bigsqcup_{U, w} G\text{Map}(G^*(U), G)_0.$$ 

- Similarly,

$$O\text{Map}(G, H)_1 \cong \bigsqcup_{U, U', w, w'} (P_{U, U'})_0.$$
Main Theorem

Theorem

When $\mathcal{G}_0/\mathcal{G}_1$ is compact,

- The inclusion $\text{OMap}_c(\mathcal{G}, \mathcal{H}) \hookrightarrow \text{OMap}(\mathcal{G}, \mathcal{H})$ is an essential equivalence.
- The groupoid $\text{OMap}_c(\mathcal{G}, \mathcal{H})$ is étale and proper.
- $\text{OMap}_c(\mathcal{G}, \mathcal{H})$ is an exponential object:

$$\text{OrbiGrpds}(\mathcal{C}^{-1})(\mathcal{G}, \text{OMap}_c(\mathcal{K}, \mathcal{H})) \cong \text{OrbiGrpds}(\mathcal{C}^{-1})(\mathcal{G} \times \mathcal{K}, \mathcal{H}).$$
Example: $\text{OMap}(*_{\mathbb{Z}/3}, S^1_{\mathbb{Z}/3})$

$S^1_{\mathbb{Z}/3} :$

\[
\begin{array}{c}
1 \mapsto e \\
1 \mapsto r \\
1 \mapsto r^2
\end{array}
\]
Example: $\text{OMap}(\ast\mathbb{Z}/3, \tilde{S}^1_{\mathbb{Z}/3})$