Lagrangian coherent structures as almost-invariant sets of a geometric heat equation

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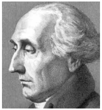
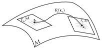
Eulerian and Lagrangian

Eulerian: physical description w/ reference to space-time



Lagrangian: physical description w/ reference to material



kinematics: description of motion of material in space

Eulerian and Lagrangian

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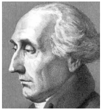

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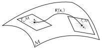

kinematics: description of motion of material in space

What describes the motion of material in Lagrangian coordinates?

Eulerian and Lagrangian

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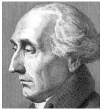

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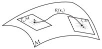

kinematics: description of motion of material in space

What describes the motion of material in Lagrangian coordinates?
The identity map!

Eulerian and Lagrangian

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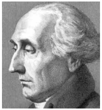

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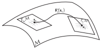

kinematics: description of motion of material in space

What describes the motion of material in Lagrangian coordinates?
The identity map!
Where has deformation gone?

Coherent Structures: Literature overview

Previous notions have diverse appearance...

- preservation of boundary length [Haller & Beron-Vera 2013] or shape [Ma & Bollt 2014]
- minimization of mixing [Froyland et al. 2010, Froyland 2013], of surface-to-volume ratio [Froyland 2015] or length of braiding material loops [Allshouse & Thiffeault 2012]
- averaging & Koopman-related methods [Mezic, Rom-Kedar, Mancho, Haller, etc.]
- many more

... but a common sense: avoiding filaments, strong stretching etc. to avoid leakage due to (weak) diffusion

Eulerian model of advection–diffusion

Fokker–Planck equation (Eulerian/spatial evolution equation):

$$\boxed{\partial_t u - \varepsilon^2 \Delta_g u = \mathrm{div}(u \cdot v)}$$

Definition (Eulerian Coherent Structures)

maximal spacetime tubes with minimal flux due to advection and diffusion discrete time/diffeo [Froyland2010,2013], continuous time [Denner, Matthes & Junge 2016]

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Observations:

1. small advective flux if tube almost follows the flow
2. small diffusive flux if surface is small

Lagrangian model of advection–diffusion

Eulerian: $$\boxed{\partial_t u - \varepsilon^2 \Delta_g u = \operatorname{div}(u \cdot v)}$$

Lagrangian "Fokker–Planck equation" (material evolution equation)

$$\boxed{\partial_t w - \varepsilon^2 \Delta_{g(t)} w = 0}$$

$$\left.\begin{array}{l} g(t) := \Phi(t)^* g\text{—pullback metric,} \\ \Delta_{g(t)}\text{—Laplace–Beltrami operator.} \end{array}\right\} \Rightarrow \text{evolving (material) manifold}$$

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Definition (Lagrangian Coherent Structures)

maximal material sets with minimal diffusive flux, or, metastable/almost-invariant sets under material evolution equation

From dynamics to geometry 1

$$\boxed{\partial_t w - \varepsilon^2 \Delta_{g(t)} w = 0}$$

Step 1: Time-dependent perturbation theory

Approximate $t \mapsto \Delta_{g(t)}$ by autonomous differential operator.

Result: $\overline{\Delta} = \frac{1}{T}\int_0^T \Delta_{g(t)}\, dt$ – dynamic Laplacian [Froyland2015]

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The dynamic Laplacian is an elliptic, nonpositive second-order differential operator. If Φ is volume-preserving, $\overline{\Delta}$ is selfajoint.

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Theorem (DK & Keller 2016, based on Lebeau & Michel 2010, cf. Froyland 2015)

$$\boxed{\mathcal{L}_\varepsilon^* \mathcal{L}_\varepsilon = I + c\varepsilon^2 \overline{\Delta} + \mathcal{O}\left(\varepsilon^4\right)} \qquad \text{(spectral convergence!)}$$

$\mathcal{L}_\varepsilon = \mathcal{D}_\varepsilon \mathcal{P}_{t_0}^{t_0+T} \mathcal{D}_\varepsilon$—*prob. TO,* $\quad \mathcal{D}_\varepsilon$—*averaging over metric balls*

From dynamics to geometry 2

$$\boxed{\partial_t w - \varepsilon^2 \overline{\Delta} w = 0}$$

Challenge: dependence on volume-preservation, and ...

Step 2: Matching principal symbols

Question: Is there an "averaged geometry" underlying the dynamic Laplacian?

From dynamics to geometry 2

$$\boxed{\partial_t w - \varepsilon^2 \overline{\Delta} w = 0}$$

Challenge: dependence on volume-preservation, and ...

Step 2: Matching principal symbols

Question: Is there an "averaged geometry" underlying the dynamic Laplacian? Not exactly, but there is a Laplace–Beltrami operator $\Delta_{\bar{g}}$ with the same principal symbol as $\overline{\Delta}$, where

$$\bar{g} := \left(\frac{1}{T} \int_0^T g(t)^{-1} \, dt \right)^{-1}$$

is the harmonic mean of pullback metrics $g(t)$.

$$\Longrightarrow \boxed{\partial_t w - \varepsilon^2 \Delta_{\bar{g}} w = 0}$$

Nice features

Geometric heat equation:

$$\boxed{\partial_t w - \varepsilon^2 \Delta_{\bar{g}} w = 0}$$

- apply theory of metastable decompositions/AI sets
- autonomous $\Rightarrow$ generator-based analysis $\Delta_{\bar{g}}$, always selfadjoint, in contrast to Eulerian generator-based approaches
- dynamics $\leftrightarrow$ operator theory $\leftrightarrow$ differential geometry
- we can visualize many different aspects of the intrinsic geometry (metric, volume density, curvature, ...)
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Thank you very much!

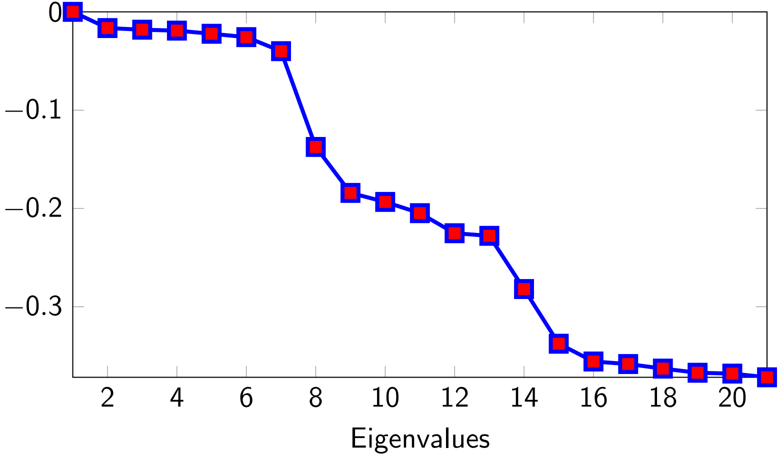
Outreach & open problems

- Conjecture: L-FPE is pullback of linearization of Eulerian Fokker-Planck equation along transport equation
- observer-independence/objectivity of Eulerian FPE
- visualization based on reference geometry, but Laplace eigenfunctions/heat flow are intrinsic (M3 homepage)
- (discrete) geometry: curvature, geodesics, Laplace operator (Note: $(M^2, \bar{g})$ not embedded in $\mathbb{R}^3$, but $\mathbb{R}^5$)
- relation to topology: LCS detection = component counting, disconnecting/decoupling M
- numerics: graph Laplacian involves geodesic distances
- applications

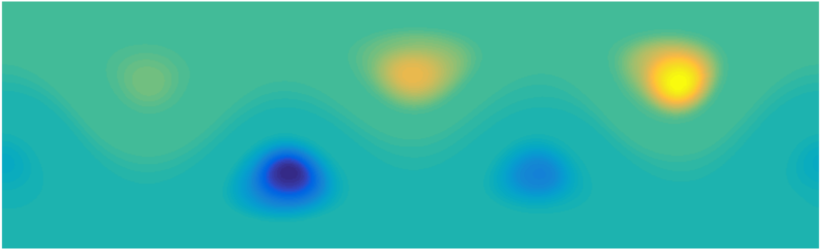
For references, manuscript, figures and videos: see my TUMpage!

Example: Bickley jet

The spectrum



Example: Bickley jet
The second eigenfunction



Cf. the second singular vector from TO calculations!