Isostatic Conjecture

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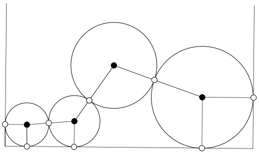
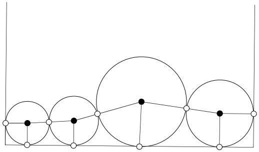
Mixing disks in a box

If you drop circular disks into a rectangular box as in the figures below, the resting configuration has to carry an equilibrium stress.

(a)  (b) 

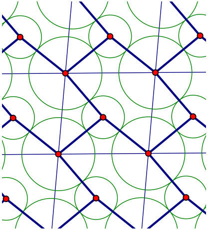
In the situation of the figure, gravity, say, is a constant force acting vertically downward. It is an easy calculation to show that there has to be at least a one-dimensional stress, and as a tensegrity framework, it has to be infinitesimally rigid, and so for n disks there has to be at least $2n$ contacts between the disks and between the disks and the fixed walls as in Figure (a). But there can be more as in Figure (b).

Isostatic Conjecture

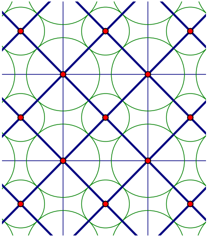
Conjecture (Isostatic Conjecture)

If circular disks packed rigidly in a "container" with generic radii, then there is only a one-dimensional equilibrium stress, and the corresponding minimal number of contacts.

Examples:



The slanted torus,
isostatic



The square torus,
not isostatic

Infinitesimal Rigidity

A tensegrity $(G, \mathbf{p})$ is a graph G, with edges labeled as bars, cables, struts, and vertices in a configuration $\mathbf{p} = (\mathbf{p_1}, \dots, \mathbf{p_n})$, where bars do not change length, cables do not increase in length, and struts do not decrease in length.
An *infinitesimal flex* $\mathbf{p}' = (\mathbf{p}'_\mathbf{1}, \dots, \mathbf{p}'_\mathbf{n})$ of a tensegrity $(G, \mathbf{p})$ is a vector $\mathbf{p}'_i$ assigned to each vertex such that

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0, \leq 0, \geq 0,$$

for a bar, cable, strut.
An *infinitesimal flex* $\mathbf{p}' = (\mathbf{p}'_\mathbf{1}, \dots, \mathbf{p}'_\mathbf{n})$ is *trivial* if it is the time 0 derivative of a one parameter family of global isometries of the ambient space. If a tensegrity $(G, \mathbf{p})$ has only trivial infinitesimal flexes, then it is called *infinitesimally rigid*.

Theorem

Infinitesimal rigidity implies local rigidity.

The Canonical Push

Depending on the container, if $\mathbf{p} = (\mathbf{p_1}, \ldots, \mathbf{p_n})$ is a configuration of centers of disks, and $\mathbf{p}' = (\mathbf{p}'_1, \ldots, \mathbf{p}'_n)$ is an infinitesimal flex of the centers, then for each disk center $\mathbf{p}_i$ define $\mathbf{p}_i(t) = \mathbf{p}_i + t\mathbf{p}'_i, t \geq 0$, and for each pair of i, j of touching disks, for $t \geq 0$,

$$(\mathbf{p}_i(t) - \mathbf{p}_j(t))^2 = (\mathbf{p}_i - \mathbf{p}_j)^2 + 2t(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) + (\mathbf{p}'_i - \mathbf{p}'_j)^2 \geq (\mathbf{p}_i - \mathbf{p}_j)^2$$

since $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) \geq 0$, and the inequality is strict unless $\mathbf{p}'_i = \mathbf{p}'_j$.

Theorem

For finite packings of disks in a flat torus, if the packing graph is rigid, then it is infinitesimally rigid.

Stresses

A *stress* for the graph, which is a scalar $\omega_{ij} = \omega_{ji}$ assigned to each edge $\{i, j\}$ of G. We say that a stress $\omega = (\ldots, \omega_{ij}, \ldots)$ is an *equilibrium stress* if for each vertex i of G the following holds

$$\sum_j \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0,$$

where $\omega_{ij} = 0$ for non-edges $\{i, j\}$. Furthermore, we say that a stress for a graph G is a *strict proper stress* if $\omega_{ij} > 0$ for a cable $\{i, j\}$, and $\omega_{ij} < 0$ for a strut $\{i, j\}$. There is no condition for a bar.

Roth-Whiteley Theorem

Theorem (Ben Roth and Walter Whiteley 1981)

A tensegrity is infinitesimally rigid if and only if the underlying bar framework is infinitesimally rigid and there is strict proper equilibrium stress.

Since infinitesimal rigidity involves the solution to a system of linear equations and inequalities, we have certain relationships among the number of vertices of a tensegrity, say n, the number of edges, e, and the dimension of the space of equilibrium stresses s. The dimension of the space of trivial infinitesimal flexes in a fixed flat torus is 2.

Proposition

For an infinitesimally rigid tensegrity on a torus $\mathbb{T}^2$ with all struts,

$$e \geq 2n - 1 \quad \textit{and} \quad s = e - (2n - 2).$$

Analytic theory of circle packings

If a packing of a torus is such that the contact graph of the packing is a triangulation of the torus, we call that packing a *triangulated packing*. The lift of the graph to the universal cover of the torus is an actual triangulation even if the graph on the torus is not an actual packing. This is also called a *compact packing* by László Fejes Toth.

Suppose that there are n circles in a triangulated packing, with e_T edges, and T triangles. Since each edge is adjacent to 2 triangles and each triangle is adjacent to 3 edges, we have

$$3T = 2e_T,$$

and by Euler's formula for the Euler characteristic of $\mathbb{T}^2$, we get that

$$n - e_T + \frac{2}{3}e_T = 0,$$

and $e_T = 3n$.

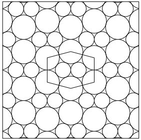
Koebe-Andreev-Thurston Theory

Theorem (Koebe-Andreev-Thurston)

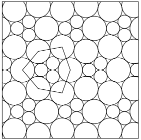
Given an abstract triangulation of a particular compact 2-manifold. Then there is a circle packing with that given contact graph in a manifold of constant curvature, and this packing is unique up to the circle preserving linear fractional transformations of the manifold.

The following torus packings are a list of compact (triangulated) packings in the plane for two disk sizes with the indicated radius ratios. These are all the ratios possible. This is by Tom Kennedy in DCG in 2006. All are known to be maximally dense packings for the indicated ratios, except for (e) and (i).

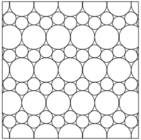
Compact Torus Packings



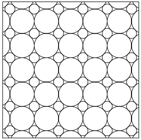
(a) $r = c_1 = 0.637556\cdots$



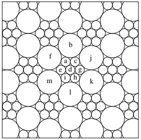
(b) $r = c_2 = 0.545151\cdots$.



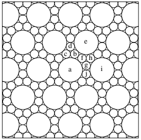
(c) $r = c_3 = 0.533296\cdots$



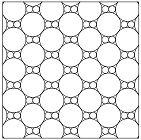
(d) $r = c_4 = 0.414214\cdots$



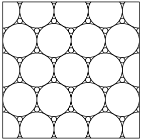
(e) $r = c_5 = 0.386106\cdots$



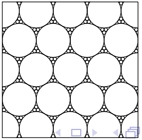
(f) $r = c_6 = 0.349198\cdots$



(g) $r = c_7 = 0.280776\cdots$



(h) $r = c_8 = 0.154701\cdots$



(i) $r = c_9 = 0.101021\cdots$

Inversive Distances

There is a way to measure the distance between pairs of non-tangent circles with disjoint interiors. We define the *inversive distance* between two circles by

$$\sigma(D_1, D_2) = \frac{|\mathbf{p}_1 - \mathbf{p}_2|^2 - (r_1^2 + r_2^2)}{2r_1 r_2},$$

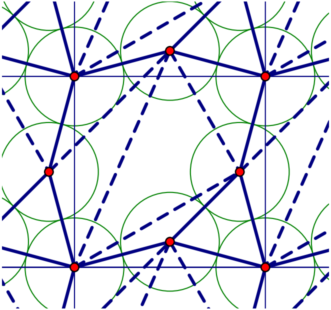
where D_1 and D_2 are disks with corresponding radii r_1 and r_2. It does not seem that there is a proof known, where any set of inversive distances determine a configuration with those inversive distances. The following local result by is enough for our purposes.

Theorem (Ren Guo)

Let $\mathcal{T}$ be a triangulation of a torus corresponding to a circle packing $\mathcal{P}$ where $\sigma(D_i, D_j) \geq 0$ is the inversive distance between each pair of disks i, j that are an edge in the triangulation $\mathcal{T}$. Then the inversive distance packings are locally determined by the values of $\sigma(D_i, D_j)$.

Completing the triangulation

Our packing graph is embedded in $\mathbb{T}^2$, and it can be completed to a triangulation by adding $e_T - e = n + 1$ additional edges. So we calculate the number of extra contacts of a triangulation. In the Figure we see an isostatic packing of 3 disks in a square torus with $2 \cdot 3 - 1 = 5$ edges, where $3 + 1 = 4$ additional edges have been inserted to create a triangulation of the torus.



The Isostatic Theorem

Theorem

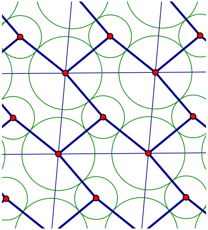
If a jammed packing $\mathcal{P}_0$ with n vertices in a torus $\mathbb{T}^2$ is chosen so that the ratio of packing disks, and torus lattice Λ_0, is generic, then the number of contacts in $\mathcal{P}_0$ is $2n - 1$, and the packing graph is isostatic.

The idea is that the dimension of the space of packings with the inversive distances replacing edge lengths is only consistent with the minimal number of contacts given by isostatic condition.

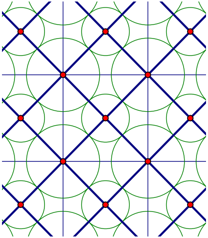
On the other hand if the lattice is held fixed, it is possible to find a "counterexample" to the isostatic conjecture.

A lame counterexample to the isostatic conjecture

These examples have 2 disks in a torus.



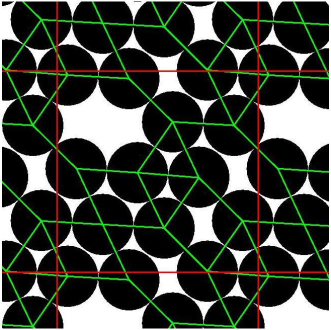
The slanted torus, isostatic



The square torus, not isostatic

Isostatic anyway

This is the conjectural most dense packing of 11 equal disks in a square torus.



Despite the specialness of the lattice and disks, it is still isostatic.

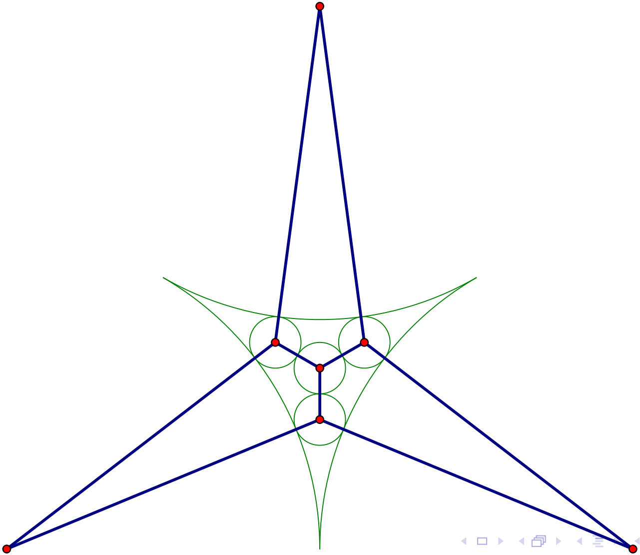
The Tricusp Case

A case when the container is a compact subset of the plane is when the boundary is three mutually tangent circular arcs. Here Guo's Theorem applies directly and there is no need to assume that the boundary is generic. Then we get:

Theorem

If a jammed packing $\mathcal{P}_0$ with n disks in a tricusp is chosen so that the ratio of the radii of the packing disks is generic, then the number of contacts in $\mathcal{P}_0$ is $2n+1$, and the packing graph is isostatic.

A 4 Disk Isostatic Tricusp



More Moving Parts

Definition

A packing of disks in a torus is *strictly jammed* if it is rigid while allowing the lattice defining the torus to move as the configuration of desk centers.

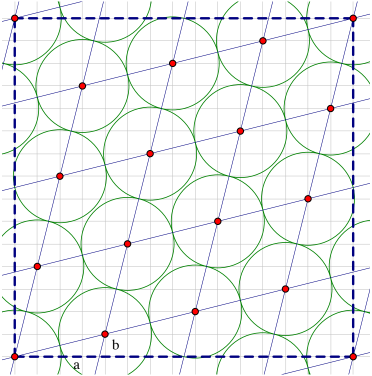
In the plane, allowing the lattice to vary provides 2 more degrees of freedom, and this requires 2 more constraints in order for the configuration to be jammed.

Theorem (Swinnerton-Dyer)

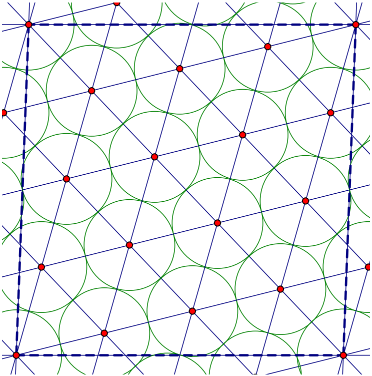
If there is an infinitesimal flex of the centers of a packing and the lattice defining the packing such that the area (volume) is not permitted to increase, then there is a finite flex moving the configuration non-trivially.

A Deformed Grid Packing

These examples have 15 disks in a torus.



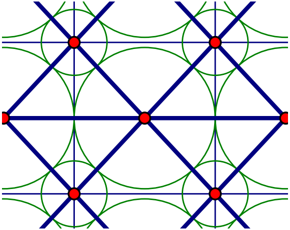
The square torus



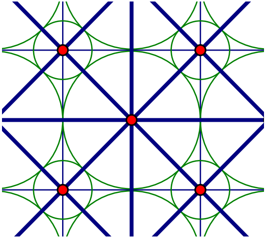
The deformed torus,
not a square lattice

Deforming the Radii

One can also deform the radii independently, not keeping the ratios constant to increase the density.



Conjectured most dense packing of two disks with ratio close to $\sqrt{2}-1$.



A most dense packing of disks with radius ratio $\sqrt{2}-1$ in the plane.

Both conjectured by László Fejes Tóth, Right example proved by Alidár Heppes.

The Deformation Game

There are three types of motions that increase the packing density. Each can be implemented with a Monty Carlo-type process, or a linear programming algorithm.

1. (Danzer) The lattice defining the torus metric is fixed while the configuration is perturbed so that the radii can be increased uniformly.
2. (Swinnerton-Dyer) The lattice is deformed decreasing its determinant (and therefore the area of the torus) adjusting the configuration while fixing the radii.
3. (Thurston) The radii are adjusted fixing the configuration and the lattice so that the packing condition is preserved while increasing the sum of the squares of the radii. This is essentially maximizing a positive definite quadratic function subject to linear constraints.

The idea is that one can perform each of these motions, separately or together depending on what is desired.