Order statistics of vectors with dependent coordinates

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based on a joint work with

K. Tikhomirov

(the paper "Order statistics of ..." available at arXiv and at http://www.math.ualberta.ca/~alexandr/)

Banff, 2017

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Given sequence of random variables $\{\xi_i\}_{i\le n}$ the sequence

$$\{k\text{-}\min_{1\le i\le n} \xi_i\}_{k\le n}$$

is the sequence of order statistics.

Order statistics in Asymptotic Geometric Analysis

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The norm $\|x\| = \sum_{j=1}^k j\text{-}\max |x_i|$, and thus, the expectation $\mathbb{E} \sum_{j=1}^k j\text{-}\max |g_i|$,

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In my work with Gordon, Schütt, and Werner, we studied norms (for a given fixed sequence a_1, ..., a_N in $\mathbb{R}^n$):

$$\|x\|_{kq} = \Big(\sum_{j=1}^{k} j\text{-}\max |\langle x, a_i \rangle|^q \Big)^{1/q}.$$

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In all such examples **maximal** order statistics appear naturally.

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Let $X = (X_1, \dots, X_n)$ be an n-dimensional random Gaussian vector with independent centered coordinates (with possibly different variances). Let T be an orthogonal transformation of $\mathbb{R}^n$ and $Y := T(X)$. Then every $k \le n$,

$$\mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{i \le n} {X_i}^2 \le \mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{i \le n} {Y_i}^2 .$$

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Our main result: this conjecture holds up to an absolute positive constant C, namely

$$\mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{i \leq n} {X_i}^2 \leq C\, \mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{i \leq n} {Y_i}^2.$$

A stronger conjecture

In their work, Mallat-Zeitouni showed that Conjecture 1 would follow from

Conjecture 2 (Mallat-Zeitouni, 2000).

Let $\{g_i\}_{i\le n}$, $\{h_i\}_{i\le n}$ be sequences of $\mathcal{N}(0,1)$ random variables such that g_i's are independent. Then for every $x \in \mathbb{R}^n$ and every $k \le n$ one has

$$\mathbb{E}\sum_{j=1}^{k} j\text{-}\min_{1\le i\le n} |x_i g_i|^2 \le \mathbb{E}\sum_{j=1}^{k} j\text{-}\min_{1\le i\le n} |x_i h_i|^2.$$

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Šidák (1967), Gluskin (1989): $\forall p > 0 \quad \mathbb{E}\max_{1\leq i\leq n} |x_i g_i|^p \geq \mathbb{E}\max_{1\leq i\leq n} |x_i h_i|^p.$

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Thus, both conjectures hold for $k = n - 1$.

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Let $f_1, \dots, f_n$ be independent copies of a random variable f with $\mathbb{E}|f| < \infty$ Let $h_1, \dots, h_n$ be copies of f. Let $x \in \mathbb{R}^n$. Then

$$26\,\mathbb{E} \sum_{j=1}^{m} j\text{-}\max_{1 \le i \le n} |x_i f_i| \ge \mathbb{E} \sum_{j=1}^{m} j\text{-}\max_{1 \le i \le n} |x_i h_i|.$$

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It turns out that it is easier to work with individual statistics than with sums.

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We say that a random variable f satisfies (α, β)-condition if every $t > 0$

$$\mathbb{P}\left(|f| \le t\right) \le \alpha t \quad \text{and} \quad \mathbb{P}\left(|f| \ge t\right) \le \exp(-\beta t).$$

Theorem 4 (Gordon, L, Schütt, Werner, 2005, 2006).

Let f_i's be independent copies of a random variable f satisfying (α, β)-condition. Let $p > 0$. Then for every $0 < x_1 \le x_2 \le \cdots \le x_n$,

$$\frac{1}{6\alpha}\left(\frac{6}{7}\right)^{1/p} \max_{1 \le j \le k} \frac{k+1-j}{\sum_{i=j}^{n} 1/x_i} \le \left(\mathbb{E}\, k\text{-}\min_{1 \le i \le n} |x_i f_i|^p\right)^{1/p}$$

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where h_i's are (dependent) copies of f. (In the Gaussian case, $C_p = \Gamma(2+p)$). This complements Šidák's result.

New results, sums

Using GLSW technique, one can prove

Theorem 5 (LT 2016).

Let f_i's be independent copies of a random variable f satisfying (α, β)-condition. Let $0 < x_1 \le x_2 \le \cdots \le x_n$. Denote

$$b_j := \sum_{i=j}^{n} 1/x_i \quad \text{and} \quad B := \sum_{j=1}^{k} \frac{(k-j+1)^p}{b_j^p}.$$

Then

$$\frac{1}{2}\left(\frac{1}{16\alpha}\right)^p B \le \mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{1\le i\le n} |x_i f_i|^p \le 3\left(\frac{4}{\beta}\right)^p \Gamma(1+p)\, B.$$

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The following claim provides simple lower bounds on quantiles.

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$$\mathbb{P}\{\xi < q(r)\} \le r \quad \text{and} \quad \mathbb{P}\{\xi \le q(r)\} \ge r.$$

The following claim provides simple lower bounds on quantiles.

Claim. *Let $k \le n$ and $0 < x_1 \le \ldots \le x_n$. For $j \le n$, set $b_j := \sum_{i=j}^n 1/x_i$. Let ξ_i, $i \le n$, be (possibly dependent) random variables satisfying the α-condition for some $\alpha > 0$, and let F_i, $i \le n$, be the distributions of $|x_i \xi_i|$. Denote*

$$F := \frac{1}{n} \sum_{i=1}^n F_i \qquad \textit{and} \qquad q := q_F\left(\frac{k - 1/2}{n}\right).$$

Then

$$q \ge \frac{1}{2\alpha} \max_{1 \le j \le k} \frac{k - j + 1}{b_j}.$$

Low bound

We need another condition. We say that the distribution F of a non-negative r.v. satisfies (A, δ)-condition for $A > 1$, $\delta \in (0, 1)$ if

$$F(t) \geq 2F(t/A) \qquad \text{whenever} \qquad F(t) \leq \delta.$$

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Theorem 6 (LT 2016).

Let $\alpha > 0$, $\delta \in (0,1)$, $A > 1$, $1 \leq k \leq n$ and $0 < x_1 \leq \ldots \leq x_n$. For $j \leq n$, set $b_j := \sum_{i=j}^{n} 1/x_i$. Further, let ξ_i, $i \leq n$, be (possibly dependent) random variables satisfying the α-condition and (A, δ)-condition. Then

$$\mathrm{Med}\left(k\text{-}\min_{1\leq i\leq n} |x_i \xi_i|\right) \geq \frac{\delta}{2A\alpha} \max_{1\leq j\leq k} \frac{k-j+1}{b_j}.$$

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Recall that Theorem 5 says that if f_i's are independent copies of a random variable f satisfying (α, β)-condition Then

$$\mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{1\leq i\leq n} |x_i f_i|^p \approx \sum_{j=1}^{k} \frac{(k-j+1)^p}{b_j^p}.$$

Comparison

Theorem 7 (LT 2016).

If f satisfies (α, β)-condition and (A, δ)-condition, f_i's are independent copies of f, h_i's are (dependent) copies of a random variable f then for all $p > 0$,

$$\mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{1 \le i \le n} |x_i f_i|^p \le 6 \left(\frac{32 A \alpha}{\delta \beta} \right)^p \Gamma(1+p) \, \mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{1 \le i \le n} |x_i h_i|^p$$

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This implies the initial Mallat-Zeitouni conjecture with an absolute constant.

Remark: Gaussian case

Let ξ_1, ..., ξ_n be standard (possibly dependent) Gaussian random variables. When

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However Van Handel's example shows that it is not true for $n = 3$.

It is natural to conjecture that the minimum attends when for all $i \neq j$,

$$\mathbb{E}\xi_i\xi_j = \frac{1}{n-1},$$

that is, when ξ_1, ..., ξ_n form a vertex set for the regular simplex in L_2.

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We show that under (A,δ)-condition, denoting

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$$F = \frac{1}{n}\sum_{i=1}^{n} F_{x_i\eta_i},$$

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$$q_G\left(\frac{k-1/2}{n}\right) \approx \mathrm{Med}\left(k\text{-}\min_{1\le i\le n}|x_i g_i|\right) \approx \sqrt{\ln k}$$

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Proof

$$\mathbb{E}|\{i \in I^c \ : \ x_i\eta_i < s\}| = \mathbb{E}\sum_{i\in I^c}\chi_{\{x_i\eta_i<s\}} \leq \sum_{i\in I^c}F_i(s)$$

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