The even dual Minkowski problem

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based on a joint works with

Károly Böröczky and Hannes Pollehn

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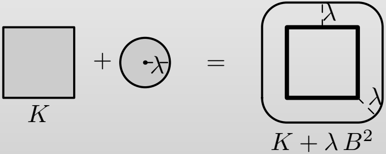
The classical Minkowski problem

- Let $K \subset \mathbb{R}^n$ be a convex body, and let B^n be the n-dimensional unit ball. The set

$$K + \lambda B^n = \{\mathbf{v} + \lambda \mathbf{w} : \mathbf{v} \in K,\ \mathbf{w} \in B^n\}$$

is the outer parallel body of K at distance λ.



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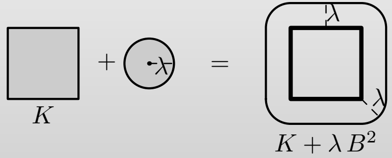

- it consists of all points $\mathbf{x}$ whose closest point $r_K(\mathbf{x})$ in K is at distance at most λ.

- Steiner's formula, 1840.

$$\mathrm{vol}\,(K + \lambda\, B^n) = \sum_{i=0}^{n} \lambda^i \binom{n}{i} W_i(K).$$

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$$W_{n-i}(K) = \frac{\operatorname{vol}(B^n)}{\operatorname{vol}_i(B^i)} \int_{G(n,i)} \operatorname{vol}_i(K|L)\,dL, \quad i = 1, \dots, n,$$

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  - $G(n,i)$ is the set of all i-dimensional linear subspaces,
    $K|L$ denotes the orthogonal projection onto L,
    $\mathrm{vol}_i(\cdot)$ denotes the i-dimensional volume.

- Let $\omega \subseteq \mathbb{S}^{n-1}$.

$$B_K(\lambda, \omega) = \left\{ \mathbf{x} \in \mathbb{R}^n : 0 < \|\mathbf{x} - r_K(\mathbf{x})\| \leq \lambda \quad \wedge \quad \frac{\mathbf{x} - r_K(\mathbf{x})}{\|\mathbf{x} - r_K(\mathbf{x})\|} \in \omega \right\}$$

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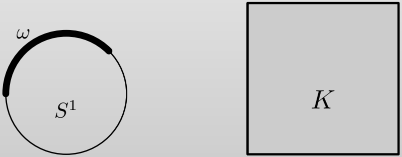

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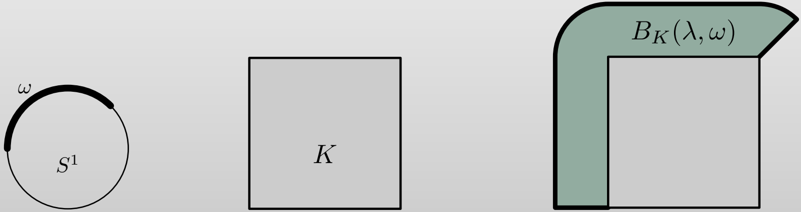

- Local Steiner formula, Fenchel&Jessen, Aleksandrov, 1938.

$$\mathrm{vol}\left(B_K(\lambda, \omega)\right) = \frac{1}{n}\sum_{i=1}^{n} \lambda^i \binom{n}{i} S_{n-i}(K, \omega),$$

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    where $\nu_K^{-1}(\omega) = \{\mathbf{x} \in \partial K : \exists \mathbf{u} \in \omega \text{ with } h_K(\mathbf{u}) = \langle \mathbf{u}, \mathbf{x}\rangle\}$, i.e., the set of boundary points of K having an outer unit normal in ω ("inverse" of the Gauß map ν_K).

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  - $S_i(K,\mathbb{S}^{n-1})=n\,W_{n-i}(K)$, $i=0,\dots,n-1$.

- Minkowski-Christoffel Problem: Characterize the area measure $S_i(K, \cdot)$ of a convex body K among all finite Borel measures μ on $\mathbb{S}^{n-1}$.

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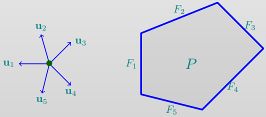

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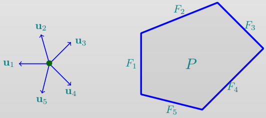

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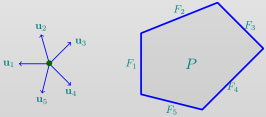

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    - $1 < i < n-1$, open.

A note on the logarithmic Minkowski problem

L_p-Brunn-Minkowski theory, Firey, 1962; Lutwak, 1993,...

- $p = 0$:

$$V_K(\omega) = \frac{1}{n} \int_{\omega} h_K(\mathbf{u})\, dS_{n-1}(K, \mathbf{u})$$

is the cone-volume measure of K.

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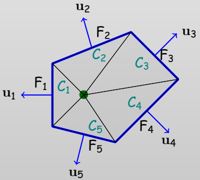
  - Let $P = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{u}_i, \mathbf{x}\rangle \leq b_i,\ 1 \leq i \leq m\}$ be a polytope with outer unit normals $\mathbf{u}_i$ and facets F_i, $1 \leq i \leq m$, and let $C_i = \operatorname{conv}(F_i \cup \mathbf{0})$ be the cone with facet F_i and apex $\mathbf{0}$.



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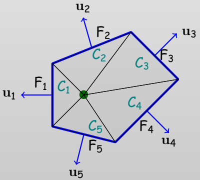

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$$V_P(\omega) = \sum_{\mathbf{u}_i \in \omega} \operatorname{vol}(C_i) = \sum_{i=1}^{m} V_P(\{\mathbf{u}_i\})\, \delta_{\mathbf{u}_i}(\omega).$$

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- Logarithmic Minkowski problem: Characterize the cone volume measure $V_K(\omega)$ of a convex body K among all finite Borel measures μ on $\mathbb{S}^{n-1}$.

- Böröczky, Lutwak, Yang, Zhang, 2013. [1] A finite *even* Borel measure μ on $\mathbb{S}^{n-1}$ is the cone-volume measure of a o-symmetric convex body if and only if it satisfies the *subspace concentration condition*,

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- i.e., for every linear subspace L holds

$$\mu(L \cap \mathbb{S}^{n-1}) \leq \frac{\dim L}{n}\,\mu(\mathbb{S}^{n-1}),$$

and equality holds for a subspace L if and only if there exists a subspace $\overline{L}$, complementary to L, such that

$$\mu(L \cap \mathbb{S}^{n-1}) + \mu(\overline{L} \cap \mathbb{S}^{n-1}) = \mu(\mathbb{S}^{n-1}).$$

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The dual Minkowski problem

Dual Brunn-Minkowski theory Lutwak, 1975,...

- For a convex body K with $\mathbf{0} \in \operatorname{int}(K)$ let

  $$\rho_K : \mathbb{R}^n \setminus \{\mathbf{0}\} \mapsto \mathbb{R}_{\geq 0} \quad \text{with} \quad \rho_K(\mathbf{x}) = \sup\{\rho \geq 0 : \rho\,\mathbf{x} \in K\}$$

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  be its radial function.
- For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ its radial addition $\widetilde{+}$ is defined as

  $$\mathbf{x} \,\widetilde{+}\, \mathbf{y} = \begin{cases} \mathbf{x} + \mathbf{y}, & \mathbf{x}, \mathbf{y} \text{ linearly dependent,} \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

- Dual outer parallel body

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\begin{aligned}
K \,\widetilde{+}\, \lambda B^n &= \left\{ \mathbf{v} \,\widetilde{+}\, \lambda \mathbf{w} : \mathbf{v} \in K,\ \mathbf{w} \in B^n \right\} \\
&= \{ \mathbf{y} \in \mathbb{R}^n : (1 - \rho_K(\mathbf{y}))\, \|\mathbf{y}\| \le \lambda \} \\
&= K \cup \{ \mathbf{y} \in \mathbb{R}^n \setminus K : \|\mathbf{y} - \rho_K(\mathbf{y})\mathbf{y}\| \le \lambda \},
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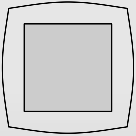

$K \widetilde{+} \lambda B^2$

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- Dual Kubota formula; Lutwak, 1979.

$$\widetilde{W}_{n-i}(K) = \frac{\operatorname{vol}(B^n)}{\operatorname{vol}_i(B^i)} \int_{G(n,i)} \operatorname{vol}_i(K \cap L)\, dL, \quad i = 1, \dots, n.$$

Huang, Lutwak, Yang, Zhang, [HLYZ], 2016.[6]

[6] *Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems*, Acta Mathematica, 216(2016)(2):325–388.

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- Let $\omega \subseteq \mathbb{S}^{n-1}$.

$$\widetilde{A}_K(\lambda, \omega) = \left\{ \mathbf{x} \in \mathbb{R}^n : (1 - \rho_K(\mathbf{x}))\, \|\mathbf{x}\| \leq \lambda,\, \rho_K(\mathbf{x})\mathbf{x} \in \nu_K^{-1}(\omega) \right\}.$$

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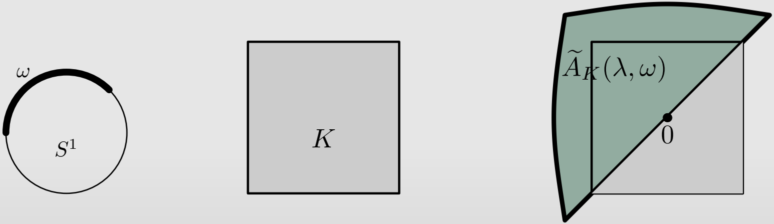

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  - $\widetilde{C}_i(K, \mathbb{S}^{n-1}) = \widetilde{W}_{n-i}(K)$.

[6] *Geometric measures in the dual Brunn-Minkowski theory and their associated Minkowski problems*, Acta Mathematica, 216(2016)(2):325–388.

- For $\omega \subseteq \mathbb{S}^{n-1}$ let

$$\begin{aligned}\alpha_K^*(\omega) &= \{\mathbf{u} \in \mathbb{S}^{n-1} : \rho_K(\mathbf{u})\mathbf{u} \in \nu_K^{-1}(\omega)\} \\ &= \mathbb{R}_{\geq 0}\, \nu_K^{-1}(\omega) \cap \mathbb{S}^{n-1}.\end{aligned}$$

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- HLYZ, 2016. Let $q \in \mathbb{R}$.

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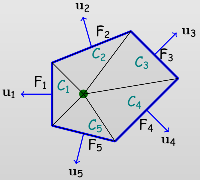

$$V_P(\omega) = \sum_{i=1}^{m} \delta_{\mathbf{u}_i}(\omega)\left(\frac{1}{n}\int_{\mathbb{R}_{\geq \mathbf{o}} F_i \cap \mathbb{S}^{n-1}} \rho_K(\mathbf{u})^q d\mathcal{H}^{n-1}(\mathbf{u})\right).$$

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  - $$\begin{aligned}\widetilde{C}_n(K, \omega) &= V_K(\omega) \text{ (cone volume measure)} \\ \widetilde{C}_0(K, \omega) &= \frac{1}{n}\mathcal{H}^{n-1}(\alpha_K^*(\omega)) \\ &\qquad \text{(Aleksandrov's integral curvature of } K^\star\text{)}\end{aligned}$$

- HLYZ, 2016. Dual Minkowski problem. Given a finite Borel measure μ on $\mathbb{S}^{n-1}$ and $q \in \mathbb{R}$. Find necessary and sufficient conditions for the existence of a convex body K (with $\mathbf{0} \in \operatorname{int} K$) such that $\widetilde{C}_q(K, \cdot) = \mu$.

- HLYZ, 2016. Let $q \in (0, n]$ A non-zero, even, finite Borel measure μ on $\mathbb{S}^{n-1}$ is the qth dual curvature measure of a o-symmetric convex body if for every proper subspace $L \subset \mathbb{R}^n$

$$\mu(\mathbb{S}^{n-1} \cap L) < \min\left\{1, \left(1 - \frac{q-1}{q}\frac{n - \dim L}{n-1}\right)\right\}\mu(\mathbb{S}^{n-1}).$$

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- For $q \in (0, 1]$ also necessary.
- For $q = n$ they coincide (up to the equality case) with the necessary and sufficient subspace concentration condition for the even logarithmic Minkowski problem.

- Böröczky, H., Pollehn, 2016 [7]. Let K be an o-symmetric convex body, $q \in (1, n)$ and let L be a proper subspace. Then

$$\widetilde{C}_q(K, \mathbb{S}^{n-1} \cap L) < \min\left\{1, \frac{\dim L}{q}\right\} \widetilde{C}_q(K, \mathbb{S}^{n-1}).$$

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- Zhao, 2016 [8]; Böröczky, LYZ, Zhao, 2017+. Let $q \in (1, n)$. A non-zero, even, finite Borel measure μ on $\mathbb{S}^{n-1}$ is the qth dual curvature measure of a symmetric convex body if for every proper subspace $L \subset \mathbb{R}^n$

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[7] Subspace concentration of dual curvature measures of symmetric convex bodies, J. Differential Geometry, accepted for publication.

[8] Existence of solution to the even dual Minkowski problem, J. Differential Geometry, accepted for publication.

- Zhao, 2016.[9] Let $q < 0$. A non-zero, even, finite Borel measure μ on $\mathbb{S}^{n-1}$ is the qth dual curvature measure of a convex body if and only if μ is not concentrated on any closed hemisphere. The convex body is uniquely determined by the measure.

[9] The dual Minkowski problem for negative indices , CVPDEs, 56(2):56:18, 2017.

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  - For $n = 2$ the bound is valid for all $q > 2$.

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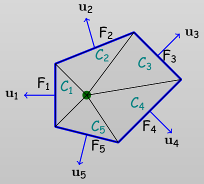
Some details...

- For $q > 0$ one may write the dual curvature measure as

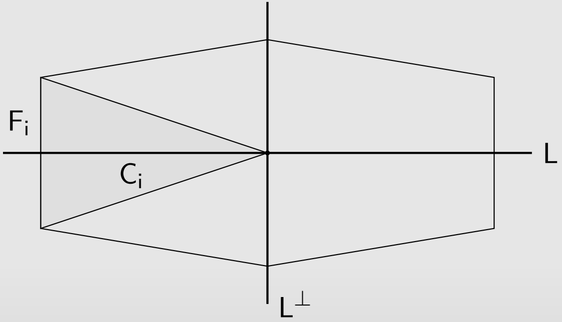
$$\widetilde{C}_q(K,\omega) = \frac{q}{n} \int_{K \cap \mathbb{R}_{\geq 0} \alpha_K^*(\omega)} \|\boldsymbol{x}\|^{q-n} \, d\mathcal{H}^n(\boldsymbol{x}),$$

i.e., for a polytope $P = \{\mathbf{x} : \langle \mathbf{x}, \mathbf{u}_i \rangle \leq b_i\}$ it is the integral of the moments $\|\cdot\|^{q-n}$ over the cones C_i with $\mathbf{u}_i \in \omega$.

$$\widetilde{C}_q(P,\omega) = \frac{q}{n} \sum_{\mathbf{u}_i \in \omega} \int_{C_i} \|\mathbf{x}\|^{q-n} \, d\mathcal{H}^n(\mathbf{x}).$$



- Each integral $\int_{C_i} \|\mathbf{x}\|^{q-n} \, d\mathcal{H}^n(\mathbf{x})$ over a cone C_i depends only on the facet F_i.



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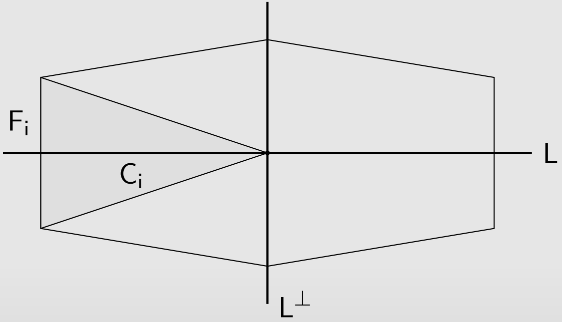

- On the other hand

$$\widetilde{C}_q(P, \mathbb{S}^{n-1}) = \frac{q}{n} \int_{P|L} \int_{P \cap (\mathbf{y}+L^\perp)} |(\mathbf{y}, \mathbf{z})|^{q-n} \, d\mathcal{H}^n(\mathbf{y}, \mathbf{z})$$

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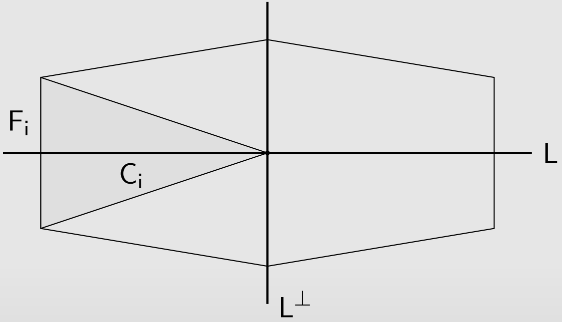

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  and the slices $P \cap (y + L^\perp)$ contain convex combinations of the form $(1-\lambda)F_i + \lambda(-F_i)$.
- Requires estimates of Brunn-Minkowski type.

- For $q \leq n$ the function $\|\cdot\|^{q-n} : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is an even quasiconcave function, i.e., the superlevel sets are o-symmetric convex sets.

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- Böröczky, H., Pollehn, 2016. Let $M \subset \mathbb{R}^n$ be a compact, convex set, $k = \dim M$ and $f \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ an $\mathcal{H}^k$-measurable, even and quasiconcave function. Then for $\lambda \in [0, 1]$

$$\int_{(1-\lambda)M+\lambda(-M)} f(\mathbf{x})\, d\mathcal{H}^k(\mathbf{x}) \geq \int_M f(\mathbf{x})\, d\mathcal{H}^k(\mathbf{x}).$$

- H., Pollehn, 2017. Let $K_0, K_1 \subset \mathbb{R}^n$ be compact, convex sets, $\dim K_0 = \dim K_1 = k \geq 1$, $\operatorname{vol}_k(K_0) = \operatorname{vol}_k(K_1)$ and their affine hulls are parallel. For $\lambda \in [0,1]$ let $K_\lambda = (1-\lambda)K_0 + \lambda K_1$. Then for $p \geq 1$

$$\int_{K_\lambda} \|\mathbf{x}\|^p \, d\mathcal{H}^k(\mathbf{x}) + \int_{K_{1-\lambda}} \|\mathbf{x}\|^p \, d\mathcal{H}^k(\mathbf{x}) \geq |2\lambda - 1|^p \left(\int_{K_0} \|\mathbf{x}\|^p \, d\mathcal{H}^k(\mathbf{x}) + \int_{K_1} \|\mathbf{x}\|^p \, d\mathcal{H}^k(\mathbf{x}) \right)$$

with equality if and only if $\lambda \in \{0,1\}$ or $p = 1$ and...

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- Corollary. Let $q \geq n+1$ and let $M \subset \mathbb{R}^n$ be a compact, convex set, $k = \dim M$. Then for $\lambda \in [0,1]$

$$\int_{(1-\lambda)M+\lambda(-M)} \|\mathbf{x}\|^{q-n} \, d\mathcal{H}^k(\mathbf{x}) \geq |2\lambda - 1|^{q-n} \int_M \|\mathbf{x}\|^{q-n} \, d\mathcal{H}^k(\mathbf{x}).$$

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  - $$\|\mathbf{x}\|^{p} = c(p, n) \cdot \int_{\mathbb{S}^{n-1}} |\langle \mathbf{x}, \mathbf{u} \rangle|^{p} \, d\mathcal{H}^{n-1}(\mathbf{u}),$$

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  - H. Kneser, Süss 1929. Inductive proof of Brunn-Minkowski inequality.
  - Karamata 1932; Hardy, Littlewood, Pólya, 1929.. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $\mathbf{x} \geq_{\text{majorizing}} \mathbf{y}$, and let $f : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, convex function. Then
    $$f(x_1) + \cdots + f(x_k) \geq f(y_1) + \cdots + f(y_k).$$

Thank you for your attention!