POSITIVE CATALYTIC AND NON-CATALYTIC POLYNOMIAL SYSTEMS OF EQUATIONS

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Lattice Walks at the Interface of Algebra, Analysis and Combinatorics

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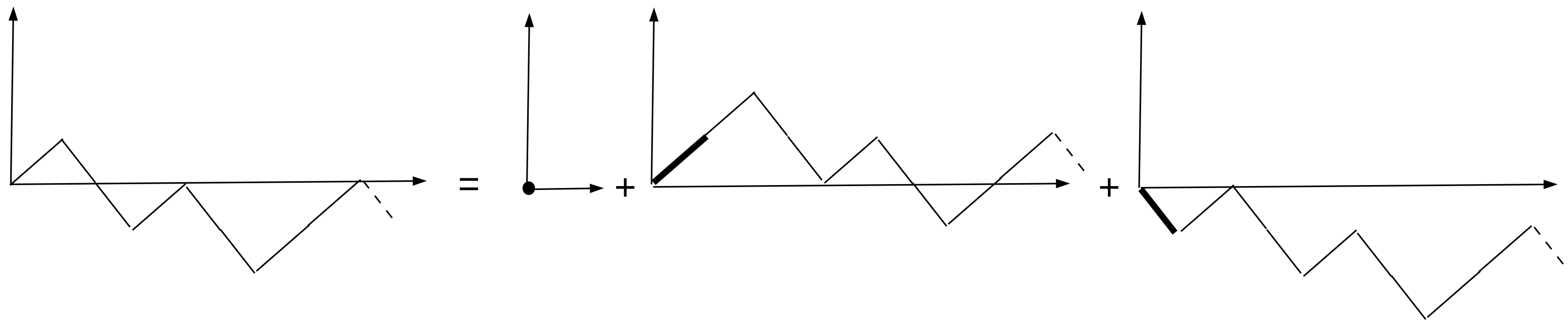
One Functional Equation

Unrestricted paths



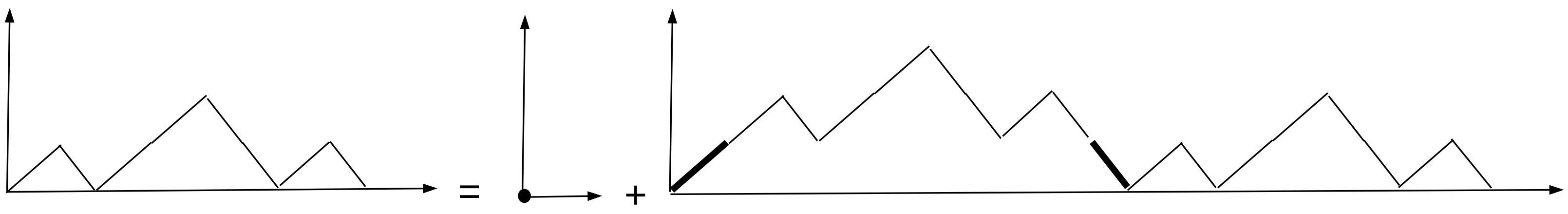
$$\boxed{B(z) = 1 + 2zB(z)}$$

$$B(z) = \frac{1}{1-2z} \quad \text{(polar singularity)}$$

$$\boxed{b_n = [z^n]B(z) = 2^n}$$

One Functional Equation

Dyck paths



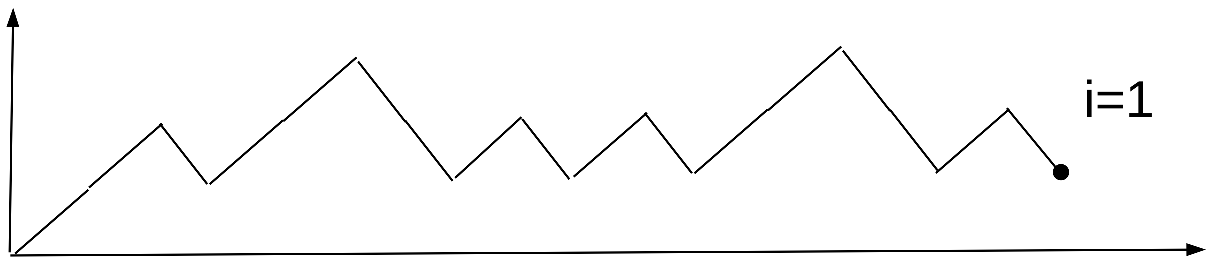
$$\boxed{B(z) = 1 + z^2 B(z)^2}$$

$$B(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z^2} \quad \text{(squareroot singularity)}$$

$$\boxed{b_{2n} = [z^{2n}]B(z) = \frac{1}{n}\binom{2n}{n} \sim \sqrt{\frac{8}{\pi}} n^{-3/2} 2^n}$$

One Functional Equation

Non-negative lattice paths



$f_{n,i}$ ... number of non-negative paths from $(0,0) \to (n,i)$

$$f_i(z) = \sum_{n \ge 0} f_{n,i} z^i \qquad F(z,u) = \sum_{i \ge 0} f_i(z) u^i = \sum_{n,i \ge 0} f_{n,i} z^n u^i$$

$$\begin{aligned} f_0(z) &= 1 + z f_1(z), \\ f_i(z) &= z f_{i-1}(z) + z f_{i+1}(z) \quad (i \ge 1) \end{aligned}$$

$$\boxed{F(z,u) = 1 + zuF(z,u) + z\frac{F(z,u) - F(z,0)}{u}}$$

u ... **“catalytic variable”**

One Functional Equation

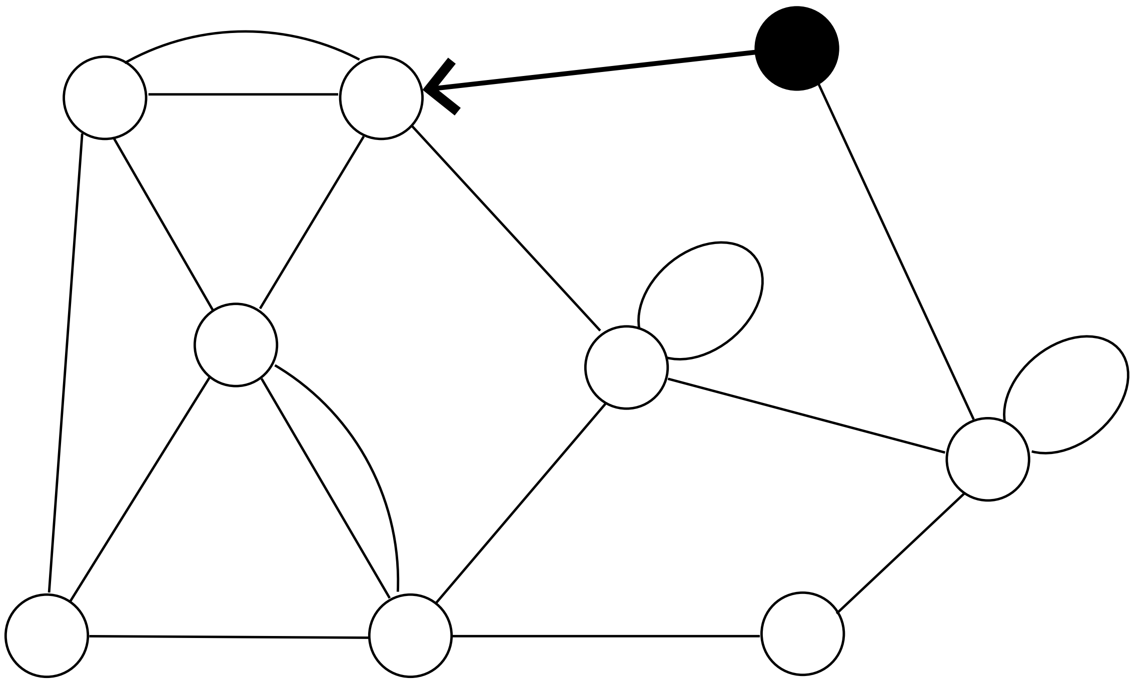
Non-negative lattice paths

$$F(z,0) = \frac{1-\sqrt{1-4z^2}}{2z^2} \quad \text{(squareroot singularity)}$$

$$\boxed{f_{2n,0} = [z^{2n}]F(z,0) = \frac{1}{n}\binom{2n}{n} \sim \sqrt{\frac{8}{\pi}}n^{-3/2}2^n}$$

One Functional Equation

Planar Maps



$M_{n,k}$ ... number of planar maps with n edges and outer face valency k

$$M(z,u) = \sum_{n,k} M_{n,k} z^n u^k$$

One Functional Equation

Planar Maps

$$\boxed{M(z,u) = 1 + zu^2 M(z,u)^2 + uz\frac{uM(z,u) - M(z,1)}{u-1}.}$$

u ... **"catalytic variable"**

$$M(z,1) = -\frac{1}{54z^2}\left(1 - 18z - \boxed{(1-12z)^{3/2}}\right) \qquad \text{(3/2-singularity)}$$

$$\boxed{M_n = [z^n]M(z,1) = \frac{2(2n)!}{(n+2)!n!}3^n \sim \frac{2}{\sqrt{\pi}} \cdot n^{-5/2} 12^n}$$

One Functional Equation

One positive linear equation

Theorem 1. *Polar singularity*:

$Q_0(z)$, $Q_1(z)$ ... polynomials with non-negative coefficients.

$$\boxed{B(z) = Q_0(z) + zQ_1(z)B(z)}$$

$$\Longrightarrow \quad \boxed{b_n = [z^n]B(z) \sim c_j \cdot z_0^{-n}}, \quad n \equiv j \bmod m$$

for $j \in \{0, 1, \ldots, m-1\}$ and some $m \geq 1$.
$z_0 > 0$ is given by $z_0 Q_1(z_0) = 1$.

Remark. Proof is simple analysis of $B(z) = Q_0(z)/(1 - zQ_1(z))$.

One Functional Equation

One positive non-linear equation

Theorem 2. [Bender, Canfield, Meir+Moon, ...] *Squareroot sing.*:

$Q(z,y)$ ... polynomial with non-negative coefficients and $Q(0,0) = 0$ and $Q_{yy} \neq 0$.

$$\boxed{B(z) = Q(z, B(z))}$$

$$\Longrightarrow \quad \boxed{b_n = [z^n]B(z) \sim c \cdot n^{-3/2} z_0^{-n}.}, \qquad n \equiv j_0 \bmod m,$$

and $\boxed{b_n = 0}$ for $n \not\equiv j_0 \bmod m$, where $m \geq 1$.
$z_0 > 0$ satisfies $b_0 = Q(z_0, b_0)$ and $1 = Q_y(z_0, b_0)$ for some $b_0 > 0$.

Remark. Proof based on the squareroot singularty
$B(z) = g(z) - h(z)\sqrt{1 - z/z_0}$ at $z = z_0$.

One Functional Equation

One positive linear catalytic equation

Theorem 3. [D.+Noy+Yu] *Squareroot singularity*:

$Q_0(z,u), Q_1(z,u), Q_2(z,u)$ ... polynomials with non-negative coefficients such that $Q_{1,u} \neq 0$ and $u \not| Q_2$.

$$\boxed{F(z,u) = Q_0(z,u) + zF(z,u)Q_1(z,u) + z\frac{F(z,u) - F(z,0)}{u}Q_2(z,u)}$$

$$\Longrightarrow \quad \boxed{f_n = [z^n]F(z,0) \sim c \cdot n^{-3/2} z_0^{-n}.}, \qquad n \equiv j_0 \bmod m,$$

(for some constants $c, z_0 > 0$) and $\boxed{f_n = 0}$ for $n \not\equiv j_0 \bmod m$, where $m \geq 1$.

One Functional Equation

One positive non-linear catalytic equation

Theorem 4. [D.+Noy+Yu] 3/2-*Singularity*:

$Q(y_0, y_1, z, u)$ ... polynomial with non-negative coefficients that is **non-linear** in y_0, y_1 (and depends on y_0, y_1) and $Q_0(u)$ a non-negative polynomial in u.

$$\boxed{M(z,u) = Q_0(u) + zQ\left(M(z,u), \frac{M(z,u) - M(z,0)}{u}, z, u\right)}$$

$$\Longrightarrow \quad \boxed{M_n = [z^n]\, M(z,0) \sim c \cdot n^{-5/2} z_0^{-n}.}, \qquad n \equiv j_0 \bmod m,$$

(for some constants $c, z_0 > 0$) and $\boxed{M_n = 0}$ for $n \not\equiv j_0 \bmod m$, where $m \geq 1$.

System of Functional Equations

$Q_1, \ldots Q_d$ ... polynomials with **non-negative** coefficients.
$y_1 = y_1(z), \ldots, y_d = y_d(z)$ ... solution of the system:

$$\begin{aligned} y_1 &= Q_1(z, y_1, \ldots, y_d), \\ &\vdots \\ y_d &= Q_d(z, y_1, \ldots, y_d). \end{aligned}$$

Recall that if $\boxed{d = 1}$ then the single equation $\boxed{y = Q(z, y)}$ has either a **polar singularity** (if it is linear) or a **squareroot singularity** (if it is non-linear).

Question. *What happends for* $\boxed{d > 1}$ *??*

System of Functional Equations

Example.

$$
\begin{aligned}
y_1 &= z(y_2 + y_1^2)\\
y_2 &= z(y_3 + y_2^2)\\
y_3 &= z(1 + y_3^2)
\end{aligned}
$$

$$
\begin{aligned}
y_1(z) &= \frac{1-(1-2z)^{1/8}\sqrt{2z\sqrt{2z\sqrt{1+2z}+\sqrt{1-2z}}+(1-2z)^{3/4}}}{2z}\\
y_2(z) &= \frac{1-(1-2z)^{1/4}\sqrt{2z\sqrt{1+2z}+\sqrt{1-2z}}}{2z}\\
y_3(z) &= \frac{1-\sqrt{1-4z^2}}{2z}
\end{aligned}
$$

$y_1(x)$ has dominant singularity $(1-2z)^{1/8}$ and $[z^n]y_1(z) \sim c\, n^{-\frac{1}{8}-1}2^n$.

System of Functional Equations

Example.

$$
\begin{aligned}
y_1 &= z(y_2^3 + y_1) \\
y_2 &= z(1 + 2y_2y_3) \\
y_3 &= z(1 + y_3^2)
\end{aligned}
$$

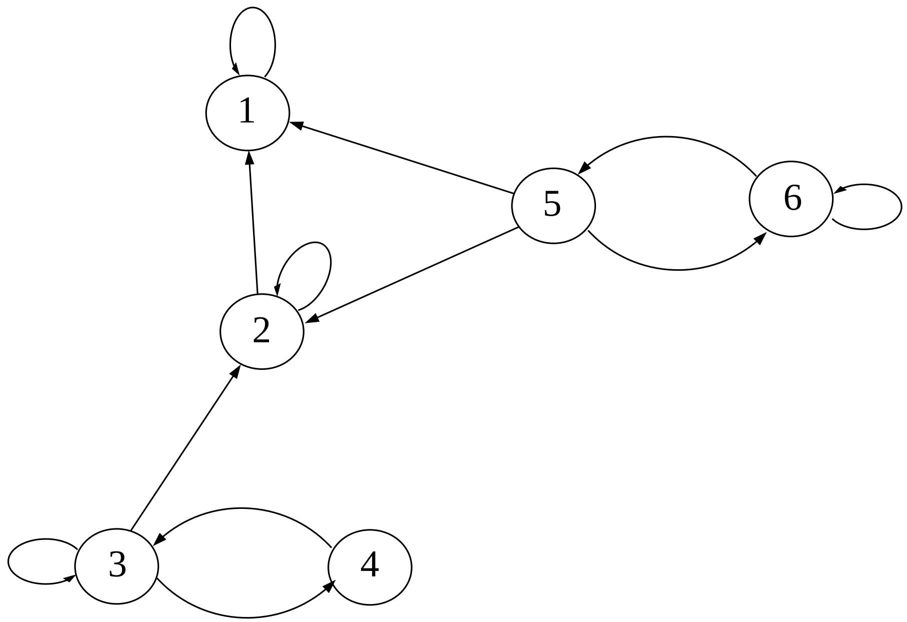
$$
\begin{aligned}
y_1(z) &= \frac{z}{1-z}\left(\frac{z}{\sqrt{1-4z^2}}\right)^3 \\
y_2(z) &= \frac{z}{\sqrt{1-4z^2}} \\
y_3(z) &= \frac{1-\sqrt{1-4z^2}}{2z}
\end{aligned}
$$

$y_1(x)$ has dominant singularity $(1-2z)^{-3/2}$ and $[z^n]y_1(z) \sim c\, n^{\frac{3}{2}-1}2^n$.

Systems of functional equations

Dependency Graph.

$$
\begin{aligned}
y_1 &= Q_1(z, y_1, y_2, y_5) \\
y_2 &= Q_2(z, y_2, y_3, y_5) \\
y_3 &= Q_3(z, y_3, y_4) \\
y_4 &= Q_4(z, y_3) \\
y_5 &= Q_5(z, y_6) \\
y_6 &= Q_6(z, y_5, y_6)
\end{aligned}
$$



Systems of functional equations

Strongly connected dependency graph

Theorem 5 [D., Lalley, Woods]

$\boxed{\mathbf{y} = \mathbf{Q}(z, \mathbf{y})}$ ... non-negative (and well defined) polynomial system of $d \ge 1$ equations such that the dependency graph is **strongly connected**.

Then the situation is the **same as for a single equation**.

It the system is **linear** then we have a common **polar singularity** and

$$\boxed{[z^n]y_1(z) \sim c_j \cdot z_0^{-n}}, \qquad n \equiv j \bmod m$$

whereas if it is **non-linear** then we have a squareroot singularity and

$$\boxed{[z^n]y_1(z) \sim c \cdot n^{-3/2} z_0^{-n}.}, \qquad n \equiv j_0 \bmod m.$$

Systems of functional equations

General dependency graph

Theorem 6 [Banderier+D.]

$\boxed{\mathbf{y} = \mathbf{Q}(z, \mathbf{y})}$ ... non-negative (and well defined) polynomial system of equations.

$$\Longrightarrow \quad \boxed{[z^n]\, y_1(z) \sim c_j\, n^{\alpha_j}\, \rho_j^{-n}} \qquad (n \equiv j \bmod m),$$

for $j \in \{0, 1, \ldots, m-1\}$ for some $m \ge 1$, where

$$\boxed{\alpha_j \in \{-2^{-k} - 1 : k \ge 1\} \cup \{m2^{-k} - 1 : m \ge 1, k \ge 0\}}.$$

Theorem 3: Kernel Method

$$F(z,u) = Q_0(z,u) + zF(z,u)Q_1(z,u) + z\frac{F(z,u) - F(z,0)}{u}Q_2(z,u)$$

rewrites to

$$F(z,u)\boxed{\left(1 - zQ_1(z,u) - \frac{z}{u}Q_2(z,u)\right)} = Q_0(z,u) - \frac{z}{u}F(z,0)Q_2(z,u).$$

If $u = u(z)$ satisfies the **kernel equation**

$$1 - zQ_1(z,u(z)) - \frac{z}{u(z)}Q_2(z,u(z)) = 0$$

Then the right hand side is also zero and we obtain

$$F(z,0) = \frac{Q_0(z,u(z))}{1 - zQ_1(z,u(z))}$$

Theorem 3: Kernel Method

The **kernel equation**

$$1 - zQ_1(z,u(z)) - \frac{z}{u(z)}Q_2(z,u(z)) = 0$$

rewrites to

$$\boxed{u(z) = zQ_2(z,u(z)) + zu(z)Q_1(z,u(z))}$$

By Theorem 2 we, thus, obtain a **squareroot singularity** for $u(z)$ which implies a **squareroot singularity** for

$$F(z,0) = \frac{Q_0(z,u(z))}{1 - zQ_1(z,u(z))}.$$

Theorem 4: Bousquet-Melou–Jehanne Method

Let $P(x_0, x_1, z, u)$ be an analytic function such that $(y(z) = M(z, 0))$

$$\boxed{P(M(z,u), y(z), z, u) = 0.}$$

By taking the derivative with respect to u we get

$$P_{x_0}(M(z,u), y(z), z, u)\, M_u(z,u) + P_u(M(z,u), y(z), z, u) = 0.$$

Key obervation:

$$\boxed{\exists\, u(z) : P_{x_u}(M(z,u(z)), y(z), z, u(z)) = 0 \Longrightarrow P_u(M(z,u(z)), y(z), z, u(z)) = 0}$$

Thus, with $f(z) = M(z, u(z))$ we get the system for $f(z), y(z), u(z)$

$$\begin{aligned} P(f(z), y(z), z, u(z)) &= 0 \\ P_{x_0}(f(z), y(z), z, u(z)) &= 0 \\ P_u(f(z), y(z), z, u(z)) &= 0. \end{aligned}$$

Theorem 4: Bousquet-Melou–Jehanne Method

Set (as given in our case)

$$\boxed{P(x_0, x_1, z, v) = F(v) + zQ(x_0, (x_0 - x_1)/v, z, v) - x_0.}$$

Then the system $P = 0$, $P_{x_0} = 0$, $P_v = 0$ rewrites to

$$f(z) = F(v(z)) + zQ(f(z), w(z), z, v(z)),$$
$$v(z) = zv(z)Q_{y_0}(f(z), w(z), z, v(z)) + zQ_{y_1}(f(z), w(z), z, v(z)),$$
$$w(z) = F_v(v(z)) + zQ_v(f(z), w(z), z, v(z)) + zw(z)Q_{y_0}(f(z), w(z), z, v(z)),$$

where

$$w(z) = \frac{f(z) - y(z)}{v(z)}.$$

This is a **positive strongly connected polynomial system**.

Theorem 4: Bousquet-Melou–Jehanne Method

Thus, by the **Theorem 5** the solution functions $f(z), v(z), w(z)$ have a **squareroot singularity** at some common singularity z_0:

$$f(z) = g_1(z) - h_1(z)\sqrt{1 - \frac{z}{z_0}},$$
$$v(z) = g_2(z) - h_2(z)\sqrt{1 - \frac{z}{z_0}},$$
$$w(z) = g_3(z) - h_3(z)\sqrt{1 - \frac{z}{z_0}}.$$

$\Longrightarrow$ $y(z) = f(z) - v(z)w(z)$ has also a squareroot singularity at z_0

$$y(z) = g_4(z) - h_4(z)\sqrt{1 - \frac{z}{z_0}} = a_0 + a_1\sqrt{1 - \frac{z}{z_0}} + a_2\left(1 - \frac{z}{z_0}\right) + a_3\left(1 - \frac{z}{z_0}\right)^{3/2} + \cdots$$

but maybe there are **cancellations of coefficients** a_j (**and actually this happens!!!**): we have $\boxed{a_1 = 0}$ and $\boxed{a_3 > 0}$.

Thank You!