Lattice walks on the half line

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Lattice Walks at the Interface
of Algebra, Analysis, and Combinatorics

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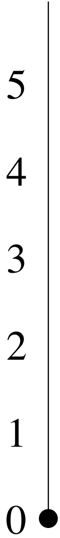
Walks on the nonnegative integers

We want to study walks on the nonnegative integers.



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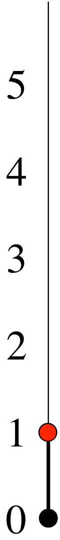

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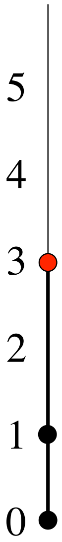

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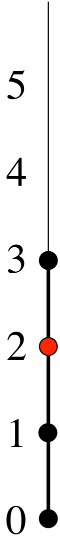

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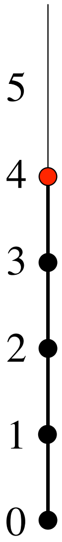
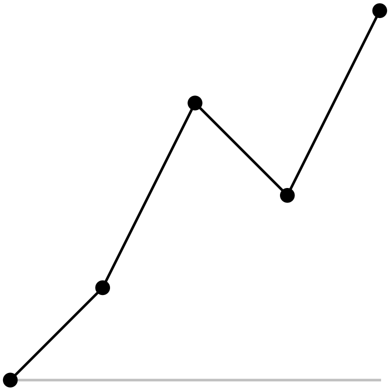


To make the walks easier to see, we give each step a horizontal component—we replace the one-dimensional step i with the two-dimensional step $(1, i)$:



An algebraicity theorem

We want to count walks on the nonnegative integers $\mathbb{N}$ with integer steps. For a set W of walks, we define its generating function to be $\sum_{w \in W} z^{\mathrm{st}(w)}$, where $\mathrm{st}(w)$ is the number of steps in w. (We could use more general weights on the steps.)

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First proved in 1980 (G).

Monsky's proof

I will discuss primarily a little-known proof of this theorem by Paul Monsky, *Generating functions attached to some infinite matrices*, Electronic J. Combin. 18 (2011), #P5.

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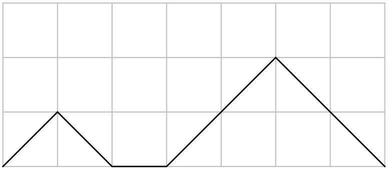
Monsky was interested in this problem because of applications to Hilbert-Kunz theory.

Monsky reduces the problem to that of counting Motzkin paths, with weights in an arbitrary ring.

Motzkin paths

Motzkin paths (or Motzkin walks) are walks on $\mathbb{N}$ starting and ending at 0 with step set $\{-1, 0, 1\}$.



We'll call -1 a *down* step, 0 a *flat* step, and 1 an *up* step.

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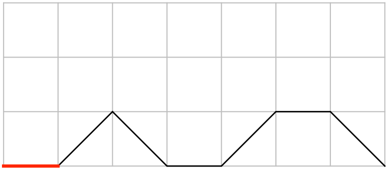

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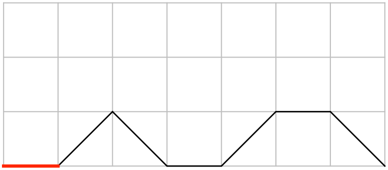
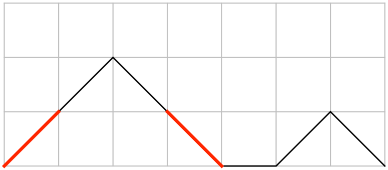


or an up step followed by a Motzkin path, then a down step, then another Motzkin path:



The functional equation

Now let us assign the weights U to an up step, F to a flat step, and D to a down step, where U, F, and D are elements of a ring (not necessarily commutative), and weight a Motzkin path by the product of the weights of its steps.

Let M be the sum of the weights of all Motzkin paths, assuming that it is summable. Then the first return decomposition gives

$$M = 1 + FM + UMDM.$$

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We may consider more general Motzkin paths in which the up, flat, and down steps come in different "colors", with different weights. Then the equation still holds where U is the sum of the weights of the up steps, F is the sum of the weights of the flat steps, and D is the sum of the weights of the down steps.

Now let S be a finite set of integers. We next show how every walk from 0 to 0 with steps in S can be reduced to a Motzkin path.

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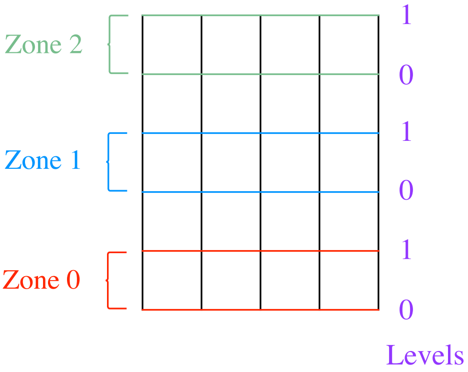
Let $t = \max\{\,|s| : s \in S\,\}$. For any integer n (which we think of as the height of a point on a path), we define its *zone* to be $\lfloor n/t \rfloor$ and its *level* to be $n \bmod t$.

So if $t = 2$, for example, we have



So to convert a path with steps in S to a Motzkin path, we contract the zones—we replace a point at height n with a point at height $\lfloor n/t \rfloor$.

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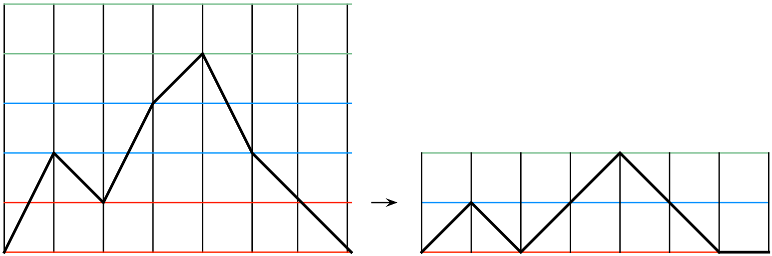
An example with $t = 2$:



Colored steps

The original path can't be recovered from the contracted Motzkin path, but if we "color" each step by its starting and ending level then we can recover the original path. We'll assign a weight of $U_{i,j}$ to an up step from level i to level j, and similarly for weights $F_{i,j}$ and $D_{i,j}$.

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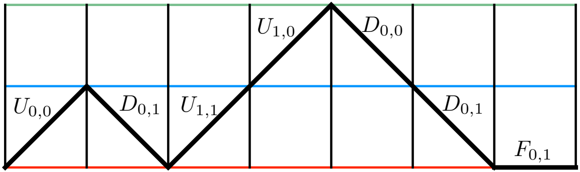

Knowing the Motzkin path and the colors of each step enables us to reconstruct the original path.

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Let $A_{i,j}$ be the $t \times t$ matrix (rows and columns indexed $0, 1, \ldots, t-1$) with a 1 in the (i, j) position and zeros elsewhere.

We take

$$U_{i,j} = F_{i,j} = D_{i,j} = A_{i,j}z.$$

Let

$$U = \sum_{j-i+t \in S} U_{i,j}, \quad F = \sum_{j-i \in S} F_{i,j}, \quad D = \sum_{j-i-t \in S} D_{i,j}.$$

Then the solution M to

$$M = 1 + FM + UMDM, \tag{$*$}$$

is a $t \times t$ matrix whose $(0,0)$ entry is the generating function for walks in $\mathbb{N}$ from $(0,0)$ with steps in S. Extracting the entries from $(*)$ gives a system of equations that shows that the entries of M are algebraic.

Example: Basketball walks

Basketball walks are walks with the step set $\{-2, -1, 1, 2\}$. They were named by A. Ayyer and D. Zeilberger (2007) and recently studied by J. Bettinelli, E. Fusy, C. Mailler, and and L. Randazzo (2016) using bijective methods, and by C. Banderier, C. Krattenthaler, A. Krinik, D. Kruchinin, V. Kruchinin, D. Nguyen, and M. Wallner (2017) using the kernel method.

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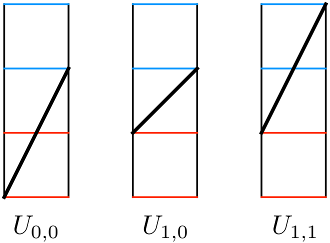
Closely related paths (essentially the same but with different weights) were counted by Jacques Labelle and Yeong-Nan Yeh (1989).

For basketball walks we have $t = 2$, so we work with 2×2 matrices. We have three types of up steps:



So

$$U = (A_{0,0} + A_{1,0} + A_{1,1})z = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} z$$

Similarly, we have

$$F = (A_{0,1} + A_{1,0})z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z$$

$$D = (A_{0,0} + A_{0,1} + A_{1,1})z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z$$

So our fundamental equation

$$M = 1 + FM + UMDM$$

becomes

$$\begin{pmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{pmatrix} z$$
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Multiplied out, this gives

$$\begin{aligned}
M_{0,0} &= 1 + zM_{1,0} + \left({M_{0,0}}^2 + M_{1,0}M_{0,0} + {M_{1,0}}^2\right) z^2 \\
M_{0,1} &= zM_{1,1} + \left(M_{0,0}M_{0,1} + M_{1,1}M_{0,0} + M_{1,1}M_{0,1}\right) z^2 \\
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\end{aligned}
$$

If we set $f = M_{0,0}$ then we can eliminate $M_{0,1}$, $M_{1,0}$, and $M_{1,1}$ to get

$$1 - (1+2z)f + (2+3z)zf^2 - (1+2z)zf^3 + z^4 f^4 = 0$$

and we find that

$$f = 1+2z^2+2z^3+11z^4+24z^5+93z^6+272z^7+971z^8+3194z^9+\cdots$$

We can express the equation for f in a simpler form: Set $g = zf$, so g satisfies

$$z - (1 + 2z)g + (2 + 3z)g^2 - (1 + 2z)g^3 + zg^4 = 0.$$

Solving for z gives

$$z = g\frac{(1 - g^2)^2}{(1 + g^3)^2}$$

So

$$g = zf = \left(z\frac{(1 - z^2)^2}{(1 + z^3)^2}\right)^{\langle -1 \rangle}$$

Other Proofs

There are three types of proofs of the algebraicity theorem that I know of.

- Factorization
- System of equations
- Formal languages

Factorization

The first proof (G., 1980) is based on a simple factorization for walks in $\mathbb{Z}$ (“Weiner-Hopf factorization"). We split the walk into three parts by cutting it at its first and last lowest point:

The middle part of this factorization gives the walks that we want to count. This leads to a factorization of the generating function

$$\frac{1}{1-\sum_{s\in S} t^s z}$$

that gives the generating function for the desired walks.

Marko Petkovšek (1998) used a different method to obtain the same type of formula.

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Cyril Banderier and Philippe Flajolet (2002) obtained the same formula using the kernel method.

Systems of equations

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Other approaches yielding a system of equations were given by Jacques Labelle and Yeong-Nan Yeh (1990), Donatella Merlini, D. G. Rogers, Renzo Sprugnoli, and M. Cecilia Verri (1999), and Philippe Duchon (2000).

Formal languages

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(2) A language accepted by a push-down automaton is context-free.

The language of nonnegative walks with a given finite set of steps is easily seen to be accepted by a push-down automaton.