Higher-order multicritical points in two-dimensional lattice polygon models

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Lattice walks at the Interface of Algebra, Analysis and Combinatorics

BIRS
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Outline

Contents

Vesicles

Vesicles are closed membranes formed of lipid bilayers

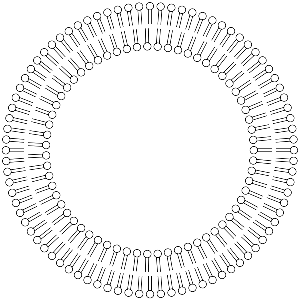

Figure: Schematic picture of a vesicle (`texample.net`).

Vesicles

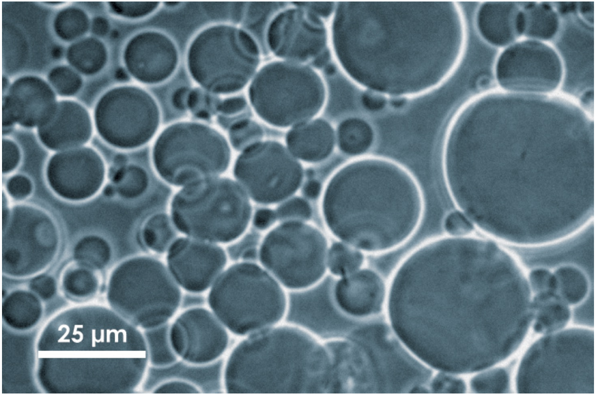

Figure: Vesicles (http://www.nanion.de).

The Fisher-Guttmann-Whittington (FGW) vesicle

We model vesicles as two-dimensional self-avoiding polygons (SAP).

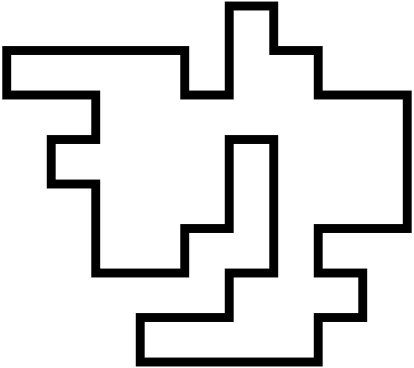

Figure: A self-avoiding polygon of perimeter 52 and area 37.

The generating function of the FGW vesicle

The area-perimeter generating function of SAP is defined as

$$\boxed{G(x,q) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{m,n}\, x^m\, q^n,}$$

where $c_{m,n}$ is the number of SAP with perimeter m and area n.

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where $c_{m,n}$ is the number of SAP with perimeter m and area n.

Conjecture (Richard, Guttmann, Jensen, 2001)

There exists a $x_c > 0$ such that for $q = e^{-\epsilon} \to 1^-$,

$$G^{\text{sing}}(x_c - s\epsilon^{\phi_c}, 1 - \epsilon) \sim \epsilon^{\theta_c} F(s),$$

where ϕ_c and θ_c are critical exponents, and $F(s)$ is called the scaling function, expressible via Airy functions.

The phase diagram of the FGW vesicle

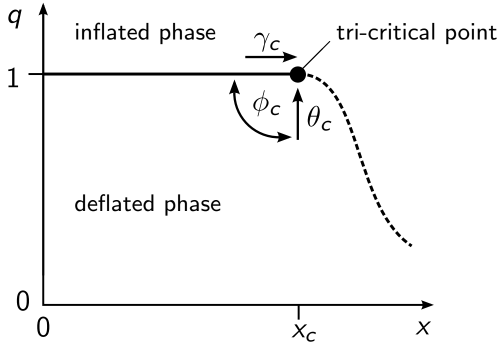

Figure: Phase diagram of the Fisher-Guttmann-Whittington vesicle.

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The model of Dyck paths

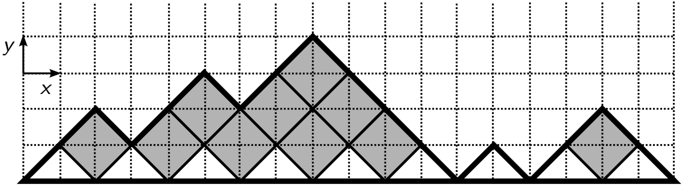

Figure: A Dyck path of half-width 9 and area 10.

We consider the generating function

$$\boxed{D(x,q) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} d_{m,n} x^m q^n,}$$

where $d_{m,n}$ is the number of DP of half-width m and area n.

Functional equation for $D(x, q)$

We have the functional equation

$$\boxed{D(x, q) = 1 + xD(qx, q)D(x, q).}$$

For $q = 1$, we get the solution

$$D(x, 1) = \frac{1}{2x}\left(1 - \sqrt{1 - 4x}\right).$$

Exact solution of $D(x, q) = 1 + xD(qx, q)D(x, q)$

Using the ansatz

$$D(x, q) = \frac{\phi(qx, q)}{\phi(x, q)},$$

we get the linearised functional equation

$$x\phi(q^2x, q) - \phi(qx, q) + \phi(x, q) = 0.$$

This equation is solved by the q-hypergeometric series

$$\phi(x, q) \;=\; {}_0\phi_1\left(\begin{matrix} - \\ 0 \end{matrix} \;; q, -x \right) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}}{(q; q)_n} (-x)^n,$$

where $(z; q)_n = \prod_{k=0}^{n-1}(1 - zq^k)$ for $z, q \in \mathbb{C}$.

Integral representation of $\phi(x,q)$

In the limit $q = e^{-\epsilon} \to 1^-$, we get

$$\phi(x,q) = A\left(\int_C \exp\left(\frac{1}{\epsilon} f(z)\right) g(z)dz\right)(1+\mathcal{O}(\epsilon)),$$

where $\epsilon = -\ln(q)$, C is a contour in the complex plane,

$$\begin{aligned} f(z) &= \log(z)\log(x) + \mathrm{Li}_2(z) - \frac{1}{2}\log(z)^2, \\ g(z) &= \sqrt{\frac{z}{1-z}}, \end{aligned}$$

and A is some function of x.

Integral representation of $\phi(x, q)$

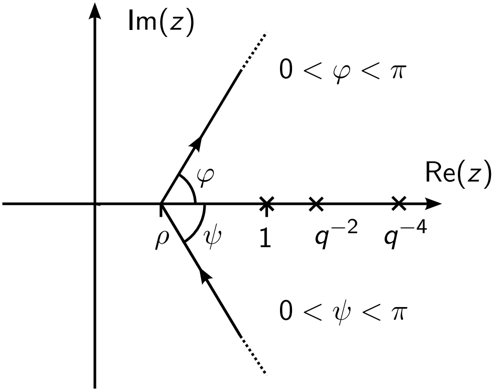

Figure: The contour C used in the integral representation of $\phi(x, q)$.

Saddle point analysis

The function $f(z)$ has the two saddle points

$$\begin{cases} z_1 &= \frac{1}{2}(1+\sqrt{1-4x}) \\ z_2 &= \frac{1}{2}(1-\sqrt{1-4x}) \end{cases}$$

which coalesce in $z_c = \frac{1}{2}$ for $x = x_c = \frac{1}{4}$.

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which coalesce in $z_c = \frac{1}{2}$ for $x = x_c = \frac{1}{4}$.

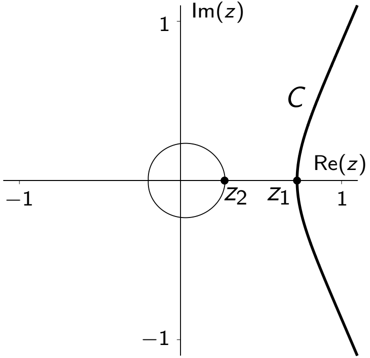
Theorem (Chester, Friedman, Ursell, 1957)

There exists a transformation $\mathrm{T} : u \mapsto z(u)$ such that

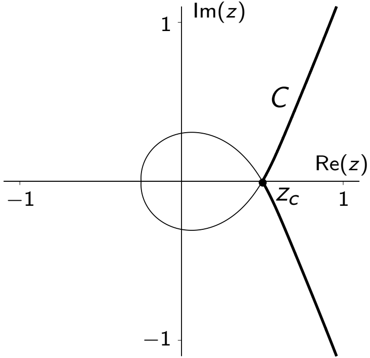
$$f(z) = \frac{1}{3}u^3 - \alpha\, u + \beta,$$

which is regular and bijective in a region containing (z_c, x_c).

Paths of steepest descent and ascent of $\mathrm{Re}(f(z))$



(a) $0 < x < x_c = \frac{1}{4}$



(b) $x = x_c = \frac{1}{4}$

Figure: Paths of steepest descent/ascent originating from $z_{1,2}$.

Uniform asymptotics of $\phi(x,q)$ and $D(x,q)$

Using the transformation $\mathsf{T} : u \mapsto z(u)$, we obtain for $q = e^{-\epsilon} \to 1^-$,

$$\phi(x,q) \sim A \int_{e^{-i\pi/3}\infty}^{e^{i\pi/3}\infty} \exp\left(\frac{1}{\epsilon}\left[\frac{u^3}{3} - \alpha\, u + \beta\right]\right) g(z(u)) \frac{dz}{du}\, du,$$

uniformly for $x > 0$.

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uniformly for $x > 0$.

Result

For $q = e^{-\epsilon} \to 1^-$

$$D(x,q) = \frac{p^{(1)}\ \mathrm{Ai}(\alpha\,\epsilon^{-\frac{2}{3}}) - q^{(1)}\epsilon^{\frac{1}{3}}\,\mathrm{Ai}'(\alpha\,\epsilon^{-\frac{2}{3}})}{p^{(0)}\ \mathrm{Ai}(\alpha\,\epsilon^{-\frac{2}{3}}) - q^{(0)}\epsilon^{\frac{1}{3}}\,\mathrm{Ai}'(\alpha\,\epsilon^{-\frac{2}{3}})} + \mathcal{O}\left(\epsilon^{\frac{2}{3}}\right)$$

uniformly for $0 < x \leq x_c = \frac{1}{4}$, where $\alpha \sim 1 - 4x$ for $x \to x_c = \frac{1}{4}$, and the $p^{(0,1)}$ and $q^{(0,1)}$ are analytic functions of x.

Scaling behaviour of $D(x, q)$

In particular, we obtain for $q = e^{-\epsilon} \to 1^-$,

$$D\left(\frac{1}{4}(1 - s\epsilon^{\frac{2}{3}}), 1 - \epsilon\right) = 2\left(1 + \epsilon^{\frac{1}{3}} F(s) + \mathcal{O}(\epsilon)\right),$$

where

$$F(s) = \frac{d}{ds} \ln(\mathrm{Ai}(s)).$$

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NH and T Prellberg.
Uniform asymptotics of area-weighted Dyck paths.
J. Math. Phys., 56:043301, 2015.

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Question

Airy function scaling is found for many models, including staircase polygons and directed column-convex polygons.

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Question (John Cardy, 2001)

How can one, by turning on further interactions, find multicritical points of higher order described by a scaling function expressible via the generalised Airy integral

$$\Theta_k(s_1, \ldots, s_{k-2}) = \frac{1}{2\pi i} \int_{e^{-i\pi/k}\infty}^{e^{i\pi/k}\infty} \exp\left(\frac{u^k}{k} - \sum_{j=1}^{k-2} s_j u^j\right) du \ ?$$

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Answer

For example by enriching the step set of Dyck paths.

Perturbation of the generating function of Dyck paths

We perturb the functional equation for the perimeter generating function $D(x) \equiv D(x,1)$ for Dyck paths with a cubic term, giving

$$wxD(x)^3 + xD(x)^2 - D(x) + 1 = 0.$$

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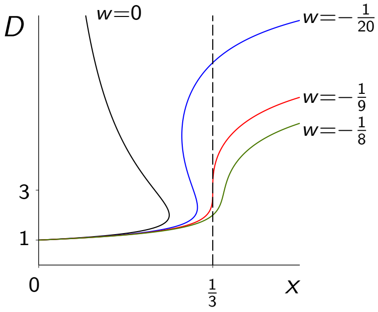

q-generalisation of the perturbed equation

We define the q-deformed version of the functional equation by

$$wxD(q^2x)D(qx)D(x) + xD(qx)D(x) - D(x) + 1 = 0,$$

where $D(x) \equiv D(w, x, q)$.

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Can this functional equation be interpreted combinatorially?

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Question

Can this functional equation be interpreted combinatorially?

Answer

Yes, the solution $D(w, x, q) \equiv D(x)$ can be interpreted combinatorially as the generating function of *deformed Dyck paths*.

The model of deformed Dyck paths

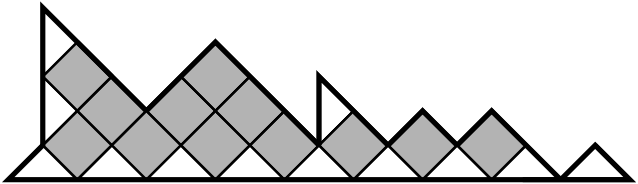


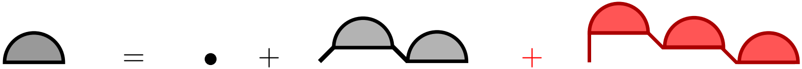
Figure: A deformed Dyck path of half-width 9, 3 jumps and area 12.

We consider the generating function

$$\boxed{D(w,x,q)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}d_{k,m,n}w^kx^mq^n,}$$

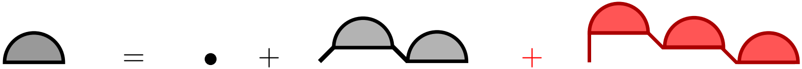
where $d_{k,m,n}$ is the number of DDP with k jumps, half-width m and area n.

Functional equation and solution



$$D(x) = 1 + xD(qx)D(x) + \color{red}{wxD(q^2x)D(qx)D(x)}$$

Functional equation and solution



$$D(x) = 1 + xD(qx)D(x) + wxD(q^2x)D(qx)D(x)$$

Analogous to Dyck paths, we obtain the solution

$$D(w,x,q) = \frac{\phi(w,qx,q)}{\phi(w,x,q)},$$

where $\phi(w,x,q) \equiv \phi(x)$ is the q-hypergeometric series

$${}_1\phi_2\left(\begin{matrix} -w \\ 0,0 \end{matrix}; q, -x\right) = \sum_{n=0}^{\infty} \frac{(-w;q)_n q^{n(n-1)}}{(q;q)_n}(-x)^n.$$

Contour integral representation of $\phi(w, x, q) = \phi(x)$

For $q = e^{-\epsilon} \to 1^-$, we get

$$\phi(x) = A \int_C \left(\exp\left(\frac{1}{\epsilon} f(z)\right) g(z) dz \right) (1 + \mathcal{O}(\epsilon)),$$

where C, is again a complex contour,

$$\begin{aligned} f(z) &= \log(z)\log(x) - \frac{1}{2}\log(z)^2 + \mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{-w}{z}\right), \\ g(z) &= \frac{z}{\sqrt{(1-z)(z+w)}} \end{aligned}$$

and A is some function of x and w.

Contour integral representation of $\phi(w, x, q)$

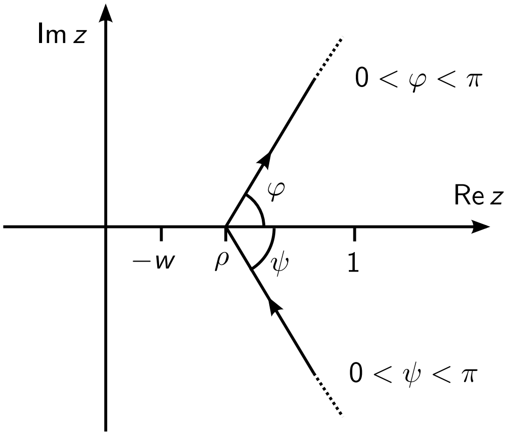
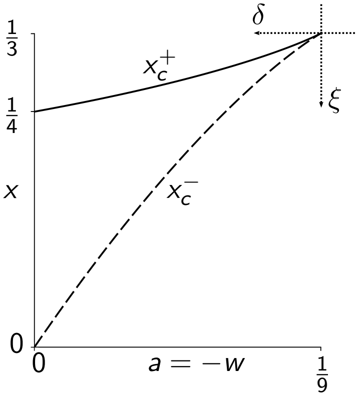


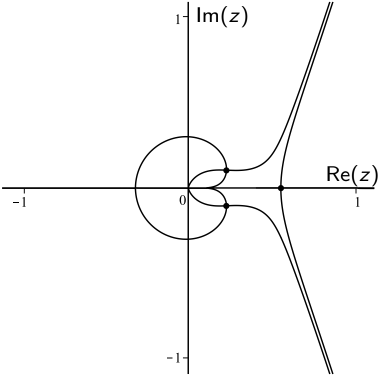
Figure: The contour C used in the integral representation of $\phi(x)$.

Saddle point analysis

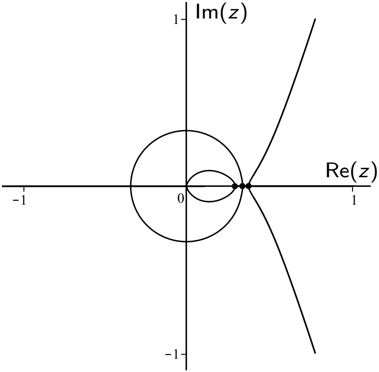
The kernel f has **three** saddle points coalescing for given w if $x = x_c^-(w)$ and $x = x_c^+(w)$. For $w = -\frac{1}{9}$, we have $x_c^- = x_c^+ = \frac{1}{3}$.



Paths of steepest descent and ascent of $\mathrm{Re}(f(z))$

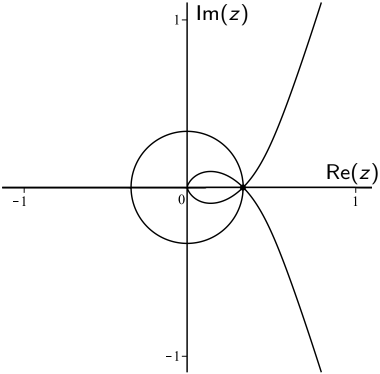


(a) $0 < x < x_c^-$

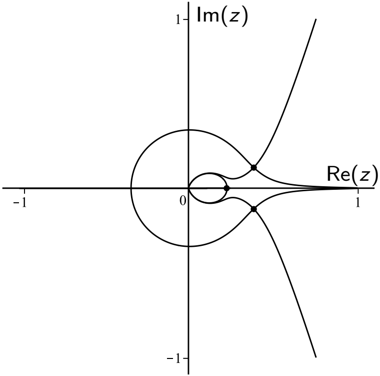


(b) $x_c^- < x < x_c^+$

Paths of steepest descent and ascent of $\mathrm{Re}(f(z))$



(c) $x = x_c^- = x_c^+$



(d) $x_c^+ < x$

Canonical transformation of f

Theorem (Ursell, 1972)

There exists a transformation $\mathrm{T} : u \mapsto z(u)$ such that

$$f(z) = \frac{1}{4}u^4 - \alpha\, u^2 - \beta u + \gamma,$$

which is regular and bijective in region containing $(z_c, x_c) = (\frac{1}{3}, \frac{1}{3})$.

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Using the transformation $\mathrm{T} : z \mapsto z(u)$, we obtain for $q = e^{-\epsilon} \to 1^-$,

$$\phi(x) \sim A \int_{e^{-i\pi/4}\infty}^{e^{i\pi/4}\infty} \exp\left(\frac{1}{\epsilon}\left[\frac{u^4}{4} - \alpha\, u^2 - \beta u + \gamma\right]\right) g(z(u)) \frac{dz}{du}\, du,$$

where A is a constant and α, β and γ are analytic functions of x and w.

Uniform asymptotics of $\phi(x)$

Define the generalised Airy function

$$\Theta(s_1, s_2) = \frac{1}{2\pi i} \int_{e^{-i\pi/4}\infty}^{e^{i\pi/4}\infty} \exp\left(\frac{u^4}{4} - s_2 u^2 - s_1 u\right) du,$$

and $\Phi(s_1, s_2) = \frac{\partial}{\partial s_1} \ln(\Theta(s_1, s_2))$.

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Theorem (NH, A Olde Daalhuis, T Prellberg 2016)

Let $q = e^{-\epsilon}$, $\delta = \mathcal{O}\left(\epsilon^{1/2}\right)$ and $\xi = \frac{3}{2}\delta + \mathcal{O}\left(\epsilon^{3/4}\right)$ as $\epsilon \to 0^+$. Then

$$G\left(\delta - \frac{1}{9}, \frac{1}{3} - \xi, q\right) = 3\left(1 + 2^{1/4}\,\Phi(s_1, s_2)\,\epsilon^{1/4} + \mathcal{O}\left(\epsilon^{1/2}\right)\right),$$

as $\epsilon \to 0^+$, for all $s_1, s_2 \in \mathbb{R}$ such that $|\Phi(s_1, s_2)| < \infty$, where $s_1 = 3\sqrt[4]{2}\left(\xi - \frac{3}{2}\delta\right)\epsilon^{-3/4}$ and $s_2 = \frac{27\sqrt{2}}{8}\left(\delta + \frac{1}{40}\xi^2\right)\epsilon^{-1/2}$.

Scaling behaviour of $D(w,x,q)$

In particular, for fixed $w=-\frac{1}{9}$, we get

$$G\left(-\frac{1}{9},\frac{1}{3}\left(1-s\epsilon^{\frac{3}{4}}\right)\right)=3\left(1+\Phi(s,0)\epsilon^{\frac{1}{4}}+\mathcal{O}(\epsilon^{\frac{1}{2}})\right),$$

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w	γ_c	θ_c	ϕ_c
$-\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{3}{4}$
$>-\frac{1}{9}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$

Table: Critical exponents of DDP.

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Table: Critical exponents of DDP.

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Higher-Order Airy Scaling in Deformed Dyck Paths.
Journal of Statistical Physics, pp. 1–16, 2017.

Numerical test

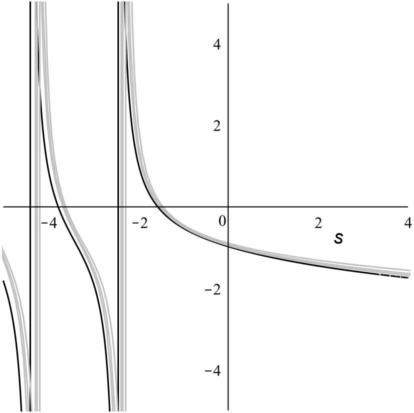

Figure: Plot of the scaling function $F\left(\sqrt[4]{2}\,s\right) = \Phi\left(\sqrt[4]{2}\,s, 0\right)$ (black) and the asymptotic approximation obtained from rearranging the scaling relation for $\epsilon = 10^{-4}, 10^{-5}, 10^{-6}$ (gray).

Contents

Higher-order multi-critical points

Generalising DDP by introducing jumps of height greater than 2, multicritical points of arbitrary order, with a multivariate scaling function expressible via the higher-order Airy function

$$\Theta(s_1, \ldots, s_n) = \frac{1}{2\pi i} \int_{e^{-\frac{i\pi}{n+2}}\infty}^{e^{\frac{i\pi}{n+2}}\infty} \exp\left(\frac{u^{n+2}}{n+2} - s_n u^n - \cdots - s_1 u\right) du,$$

can be observed.

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N Haug and T Prellberg.
Multicritical points in a two-dimensional lattice vesicle model.
In preparation.

The End.