Duality for Metaplectic Ice

arXiv:1709.06500

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Also:
Brubaker, Bump, Chinta, Friedberg, Gunnells (arXiv:1009.1741)
Brubaker, Buciumas and Bump (arXiv:1604.02206)
Brubaker, Buciumas, Bump and Friedberg (arXiv:1704.00701)

Partition functions

In statistical mechanics, we consider an ensemble of many states. If s is a state let $E(s)$ be its energy.

More energetic states become more probable if the system is at a higher temperature.

In Maxwell-Boltzmann statistics the probability of state s is proportional to $e^{-E(s)/k_B T}$ where k_B is Boltzmann's constant.

Thus the probability of state s is

$$\frac{1}{Z}\, e^{-E(s)/k_B T} \qquad \text{where} \qquad Z = \sum_s e^{-E(s)/k_B T}.$$

This Z is the partition function. Its significance goes beyond this probabilistic fact. In some sense it contains complete thermodynamic information about the system.

Solvable Lattice Models

- It was found that certain 2-dimensional lattice models can be analyzed completely.
- The first such solvable lattice model to be fully analyzed was the Ising model (Onsager, 1944).
- An identity that Onsager and Baxter called the "star-triangle relation" was used powerfully by Baxter. It was renamed the Yang-Baxter equation by Faddeev.
- For us, the Yang-Baxter equation characterizes solvability of the model.
- Fitting it into an algebraic framework led to the invention of quantum groups (Drinfeld, Jimbo).

Analytic continuation of the partition function

There is utility in considering nonreal Boltzmann weights. A spin is ± 1. A state of the Ising model consists of an assignment of spins $s = \{s_i\}$ at vertices v_i of a lattice.

$$Z = \sum_s e^{-E(s)/k_B T}, \qquad -E(s) = \sum_{i,j \text{ adjacent}} J s_i s_j + \sum_i t s_i.$$

Here t is a parameter corresponding to an external field. It may be fruitfully be regarded as a complex parameter.

Theorem

(Lee-Yang) The zeros of Z lie on the line $\mathrm{re}(t) = 0$.

This is a kind of Riemann hypothesis. (There are other similarities between partition functions and L-functions!) We do not require the Boltzmann weights to be real.

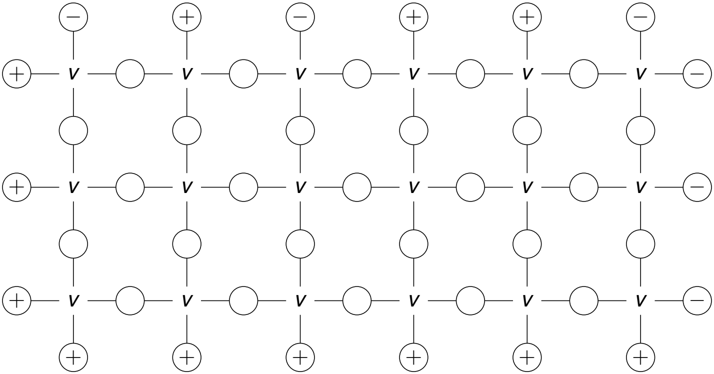
Duality in Lattice Models

- Sometimes a system may be described in very different ways.
- In quantum mechanics, the world looks different in position or momentum coordinates, related by the Fourier transform.
- Kramers and Wannier proved that for the Ising model, the partition functions of high and low temperature models are related, computing the location of the phase transition.
- In Type IIB string theory, S duality relates the partition functions for inverse coupling constant y with the system for $1/y$. The partition function is an automorphic form.
- We will describe two systems originating in number theory that have the same partition function.

Six Vertex Model

Begin with a grid, usually (but not always) rectangular:

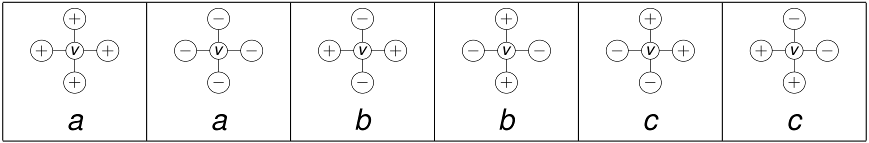
column: 5 4 3 2 1 0



Edges are labeled with “spins” $\pm$. Boundary spins are fixed. A state assigns spins to interior edges.

Boltzmann weights: 6 Vertex Model

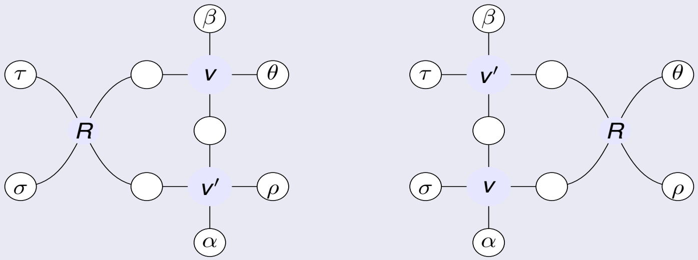
- A state of the system assigns spins to the interior edges.
- Each state has a Boltzmann weight which is the product of Boltzmann weights for each vertex.
- The Boltzmann weights for a vertex depend on the spins of the adjacent edges.
- The field-free six vertex model was analyzed by Lieb, Sutherland, Baxter, Korepin, Izergin. It was a primary example leading to the discovery of quantum groups.
- Let $a = a(v)$, $b = b(v)$, $c = c(v)$ be complex numbers. They depend on the vertex v. Let $\Delta(v) = \frac{a^2+b^2-c^2}{2ab}$.



Yang-Baxter equation: 6 vertex model

Theorem

*(**Baxter**) If $\Delta(v) = \Delta(v')$ (condition on Boltzmann weights) there is a vertex R such that for all boundary spins $\sigma, \tau, \cdots$ the partition functions*

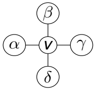


are equal. (The interior edge spins are summed.)

Implies: (Baxter) Field-free 6-vertex model is solvable.

Braided Categories

We may associate with every edge ϵ a module V_ϵ in a monoidal category. This module is a vector space with basis $+$ and $-$.



Boltzmann weights are arranged as coefficients in the R-matrix!

is interpreted as an isomorphism $V_\alpha \otimes V_\beta \to V_\gamma \otimes V_\delta$. The Yang-Baxter equation means this diagram commutes:

$$V_\tau \otimes V_\sigma \otimes V_\beta \to V_\tau \otimes V_\beta \otimes V_\sigma$$

$$V_\sigma \otimes V_\tau \otimes V_\beta \qquad\qquad V_\tau \otimes V_\sigma \otimes V_\beta$$

$$V_\sigma \otimes V_\beta \otimes V_\tau \to V_\beta \otimes V_\sigma \otimes V_\tau$$

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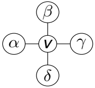

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$$V_\sigma \otimes V_\tau \otimes V_\beta \to V_\tau \otimes V_\sigma \otimes V_\beta \to V_\tau \otimes V_\beta \otimes V_\sigma \to V_\beta \otimes V_\tau \otimes V_\sigma$$

$$V_\sigma \otimes V_\tau \otimes V_\beta \to V_\sigma \otimes V_\beta \otimes V_\tau \to V_\beta \otimes V_\sigma \otimes V_\tau \to V_\beta \otimes V_\tau \otimes V_\sigma$$

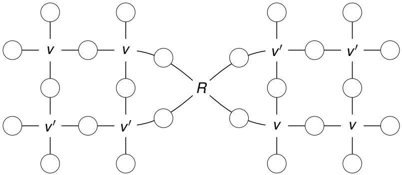
This is an example of a braided category

Quantum Groups

- Given a braided category Tannakian theory or other methods (Drinfeld, FRT, etc.) tries to produce a Hopf algebra whose (co)modules are the given category.
- For the field-free Yang-Baxter equation, we get $U_q(\widehat{\mathfrak{sl}}_2)$ where

$$\frac{1}{2}(q - q^{-1}) = \Delta = \frac{a^2 + b^2 - c^2}{2ab}.$$

- This is the historical route to quantum groups.



Metaplectic Whittaker functions as partition functions

If F is a p-adic field containing the n-th roots of unity μ_n, there is a central extension

$$1 \longrightarrow \mu_n \longrightarrow \widetilde{\mathrm{GL}}(r) \longrightarrow \mathrm{GL}(r) \longrightarrow 1.$$

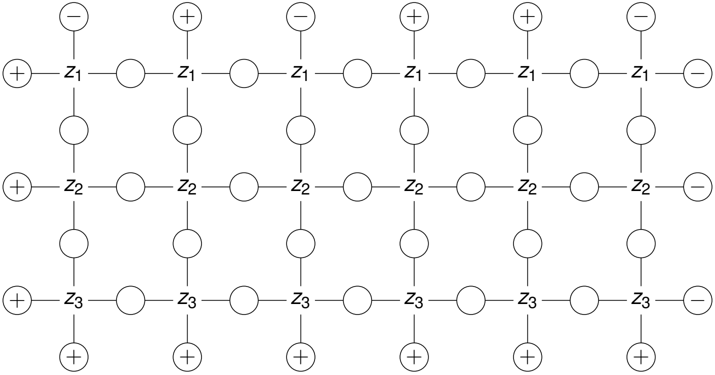
The group $\widetilde{\mathrm{GL}}(r)$ is the metaplectic group.

- Interesting automorphic forms live on metaplectic groups.
- Brubaker, Bump, Chinta, Friedberg, Gunnells proposed ...
- ... and Brubaker, Buciumas and Bump, showed:
- Whittaker functions on $\widetilde{\mathrm{GL}(r)}$ may be represented as partition functions of solvable lattice models.
- The relevant quantum group is $U_{\sqrt{q^{-1}}}(\widehat{\mathfrak{gl}}(n|1))$.
- This setup exhibits an interesting duality.

Metaplectic Ice

Proceed as before but label vertices by $z_i \in \mathbb{C}^\times$:

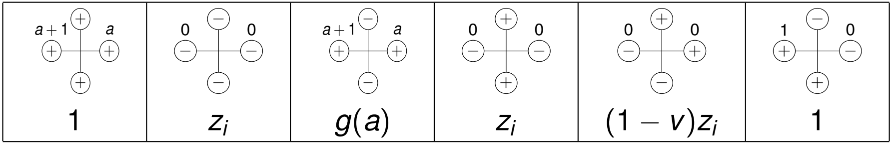
column: 5 4 3 2 1 0



Boltzmann Weights for metaplectic Γ Ice

Choose parameters $g(a)$ depending on a mod n:
$g(a)\,g(-a) = 1/v$ if $n \nmid a$,
$g(a) = -v$ if $n|a$. (Secretly Gauss sums).

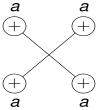
Let $z_i \in \mathbb{C}^\times$. Here are the Boltzmann weights:



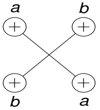
- Horizontal edges have $n+1$ possible states: $+a$ (a mod n) or -0.
- The module corresponding to a horizontal edge is a $(n|1)$-dimensional supervector space.

The ΓΓ R-matrix

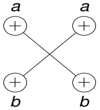
Another type of vertex, the R-matrix $R^{\Gamma\Gamma}_{z_1,z_2}$.



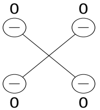
$z_2^n - v z_1^n$



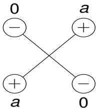
$g(a-b)(z_1^n - z_2^n)$



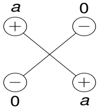
$(1-v) z_1^{a-b} z_2^{n-a+b}$



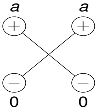
$z_1^n - v z_2^n$



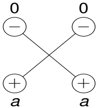
$v(z_1^n - z_2^n)$



$z_1^n - z_2^n$

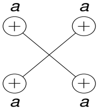
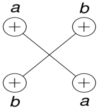
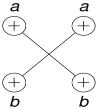
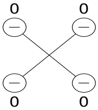
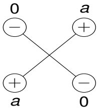
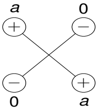
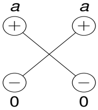
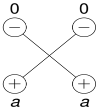


$(1-v) z_1^a z_2^{n-a}$



$(1-v) z_1^{n-a} z_2^a$

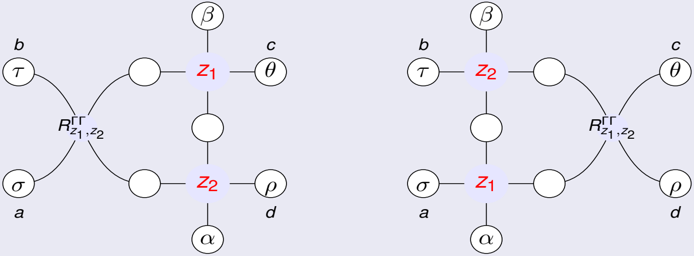
The ΓΓ R-matrix

Another type of vertex, the R-matrix $R^{\Gamma\Gamma}_{z_1,z_2}$. Drinfeld twist of $\widehat{\mathfrak{gl}}(n\|1)$	 $z_2^n - vz_1^n$	 $g(a-b)(z_1^n - z_2^n)$
 $(1-v)z_1^{a-b}z_2^{n-a+b}$	 $z_1^n - vz_2^n$	 $v(z_1^n - z_2^n)$
 $z_1^n - z_2^n$	 $(1-v)z_1^a z_2^{n-a}$	 $(1-v)z_1^{n-a}z_2^a$

Gamma-Gamma Yang-Baxter Equation

Theorem

The following partition functions are equal.

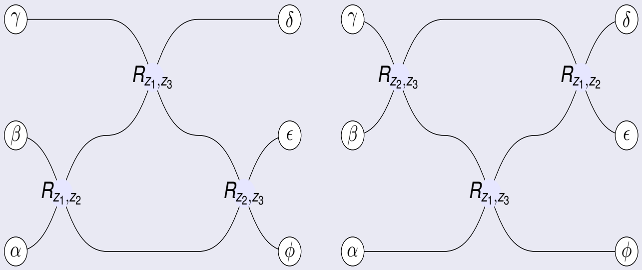


(Here a, b, c, d are the charges.)

Another Yang-Baxter equation

Theorem

Fix z_1, z_2 and z_3 and (decorated) boundary spins. The following partition functions are equal:

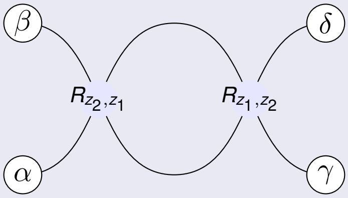


Brubaker, Buciumas, Bump and Friedberg use this to construct representations of the affine Hecke algebra.

Complementary Equation

Theorem

Let $\alpha, \beta, \gamma, \delta$ be decorated spins. Then the partition function



equals $\begin{cases} (z_1^n - vz_2^n)(z_2^n - vz_1^n) & \text{if } \alpha = \gamma,\ \beta = \delta \\ 0 & \text{otherwise.} \end{cases}$

The horizontal edges correspond to evaluation modules for a Drinfeld twist of $U_{\sqrt{v}}(\widehat{\mathfrak{gl}}(n|1))$

Whittaker coinvariants as $U_{\sqrt{q^{-1}}}(\widehat{\mathfrak{gl}}(n))$-modules

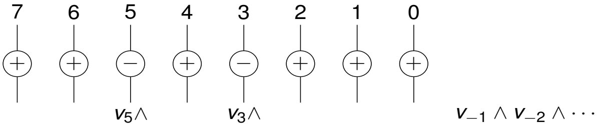
- V: A representation of $GL(r, F)$, $V_{u,\psi}$ its module of Whittaker coinvariants, the largest quotient through which every Whittaker functional factors.
- $\mathbf{z} = (z_1, \cdots, z_r) \in \mathbb{C}^r \cong \widehat{T}(\mathbb{C})$, the Langlands dual torus. It parametrizes a principal series representation $V_{\mathbf{z}}$.
- $\mathcal{A}_w : V_{\mathbf{z}} \to V_{w\mathbf{z}}$ interwining integral.

The scattering matrix of $\mathcal{A}_{s_i}$ on $(V_{\mathbf{z}})_{U,\psi}$ was computed by Kazhdan and Patterson. It agrees with the R-matrix of (Drinfeld twisted) $U_{\sqrt{q^{-1}}}(\widehat{\mathfrak{gl}}(n))$!

$$\begin{array}{ccc} (V_{\mathbf{z}})_{U,\psi} & \xrightarrow{\cong} & V_{z_1} \otimes \cdots \otimes V_{z_i} \otimes V_{z_{i+1}} \otimes \cdots \otimes V_{z_r} \\ \downarrow \mathcal{A}_{s_i} & & \downarrow \text{R-matrix} \\ (V_{s_i\mathbf{z}})_{U,\psi} & \xrightarrow{\cong} & V_{z_1} \otimes \cdots \otimes V_{z_{i+1}} \otimes V_{z_i} \otimes \cdots \otimes V_{z_r} \end{array}$$

Fermionic interpretation of vertical edges

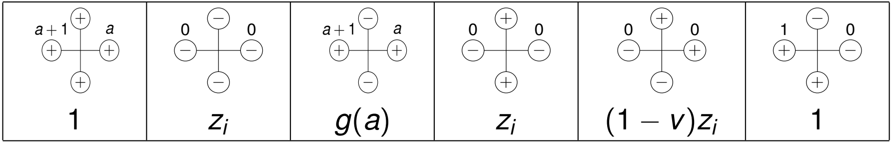
- The vertical edges which take values $\pm$ have no interpretation as for $U_{\sqrt{v}}(\widehat{\mathfrak{gl}}(n))$ modules, but the aggregation of them may be interpreted as an element of the fermionic Fock space.



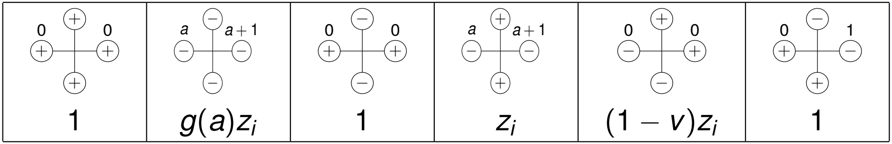
- Kashiwara, Miwa and Stern defined a $U_q(\widehat{\mathfrak{gl}}(n))$ module structure on the degree k Fermionic Fock space $\mathfrak{F}_{(k)}$ and a vertex operator $V_z \otimes \mathfrak{F}_{k-1} \longrightarrow \mathfrak{F}_k$. We think the row transfer matrix for Gamma ice computes this.

Gamma ice versus Delta Ice

Gamma Ice:



Delta Ice:



Same boundary conditions for Delta ice but reverse the rows.

Theorem (Duality Theorem)

The partition functions are equal!

Why this was discovered

- An equivalent statement was proved combinatorially by Brubaker, Bump and Friedberg (2011).
- The proof was difficult.
- Motivation: this is the *p*-part of a multiple Dirichlet series. This combinatorial fact implies the analytic continuation and functional equation.
- Brubaker, Bump, Chinta, Friedberg and Gunnells proposed that this could be proved using the Yang-Baxter equation (2012).
- A proof using the Yang-Baxter equation was given by Brubaker, Buciumas, Bump and Gray (2017)

Theorem (Duality Theorem)

The partition functions are equal!