Spectral theory for the complex Airy operator: the case of a semipermeable barrier and applications to the Bloch-Torrey equation
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The transmission boundary condition which is considered appears in various exchange problems such as molecular diffusion across semi-permeable membranes [36, 33], heat transfer between two materials [11, 18, 8], or transverse magnetization evolution in nuclear magnetic resonance (NMR) experiments [20]. In the simplest setting of the latter case, one considers the local transverse magnetization $G(x, y\,; t)$ produced by the nuclei that started from a fixed initial point y and diffused in a constant magnetic field gradient g up to time t.

This magnetization is also called the propagator or the Green function of the Bloch-Torrey equation [38] (1956):

$$\frac{\partial}{\partial t} G(x, y\,; t) = (D\Delta - i\gamma g x_1)\, G(x, y\,; t)\,, \tag{1}$$

with the initial condition

$$G(x, y\,; t = 0) = \delta(x - y), \tag{2}$$

where D is the intrinsic diffusion coefficient, $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_d^2$ the Laplace operator in $\mathbb{R}^d$, γ the gyromagnetic ratio, and x_1 the coordinate in a prescribed direction.

In this talk, we focus on the one-dimensional situation ($d = 1$), in which the operator

$$D_x^2 + ix = -\frac{d^2}{dx^2} + ix$$

is called the complex Airy operator and appears in many contexts: mathematical physics, fluid dynamics, time dependent Ginzburg-Landau problems and also as an interesting toy model in spectral theory (see [3]). We consider a suitable extension $\mathcal{A}_1^+$ of this differential operator and its associated evolution operator $e^{-t\mathcal{A}_1^+}$. The Green function $G(x, y\,; t)$ is the distribution kernel of $e^{-t\mathcal{A}_1^+}$.

For the problem on the line $\mathbb{R}$, an intriguing property is that this non self-adjoint operator, which has compact resolvent, has empty spectrum. However, the situation is completely different on the half-line $\mathbb{R}_+$. The eigenvalue problem

$$(D_x^2 + ix)u = \lambda u,$$

for a spectral pair (u, λ) with u in $H^2(\mathbb{R}_+)$, $xu \in L^2(\mathbb{R}_+)$ has been thoroughly analyzed for both Dirichlet ($u(0) = 0$) and Neumann ($u'(0) = 0$) boundary conditions.
The spectrum consists of an infinite sequence of eigenvalues of multiplicity one explicitly related to the zeroes of the Airy function (see [35, 26]).
The space generated by the eigenfunctions is dense in $L^2(\mathbb{R}_+)$ (completeness property) but there is no Riesz basis of eigenfunctions.
Finally, the decay of the associated semi-group has been analyzed in detail through Gearhard-Prüss like theorems.
The physical consequences of these spectral properties for NMR experiments have been first revealed by Stoller, Happer and Dyson [35], then by De Sviet et al. and D. Grebenkov [15, 19, 22].

In this talk, we consider another problem for the complex Airy operator on the line but with a transmission property at 0 which reads (cf Grebenkov [22]),

$$\left\{ \begin{array}{lcl} u'(0_+) & = & u'(0_-)\,, \\ u'(0) & = & \kappa\left(u(0_+) - u(0_-)\right)\,, \end{array} \right. \tag{3}$$

where $\kappa \geq 0$ is a real parameter.
The case $\kappa = 0$ corresponds to two independent Neumann problems on $\mathbb{R}_-$ and $\mathbb{R}_+$ for the complex Airy operator.
When κ tends to $+\infty$, the second relation in (3) becomes the continuity condition, $u(0_+) = u(0_-)$, and the barrier disappears.
Hence, the problem tends (at least formally) to the standard problem for the complex Airy operator on the line.

We summarize the main (1D)-results in the following:

Theorem

The semigroup $\exp(-t\mathcal{A}_1^+)$ is contracting. The operator $\mathcal{A}_1^+$ has a discrete spectrum $\{\lambda_n(\kappa)\}$. The eigenvalues $\lambda_n(\kappa)$ are simple and determined as (complex-valued) solutions of the equation

$$2\pi \mathrm{Ai}'(e^{2\pi i/3}\lambda)\mathrm{Ai}'(e^{-2\pi i/3}\lambda) + \kappa = 0, \tag{4}$$

where $\mathrm{Ai}'(z)$ is the derivative of the Airy function.
For all $\kappa \geq 0$, there exists N such that, for all $n \geq N$, there exists a unique eigenvalue of $\mathcal{A}_1^+$ in the ball $B(\lambda_n^\pm, 2\kappa|\lambda_n^\pm|^{-1})$, where $\lambda_n^\pm = e^{\pm 2\pi i/3}a_n'$, and a_n' are the zeros of $\mathrm{Ai}'(z)$.
Finally, for any $\kappa \geq 0$ the space generated by the eigenfunctions of the complex Airy operator with transmission is dense in $L^2(\mathbb{R}_-) \times L^2(\mathbb{R}_+)$.

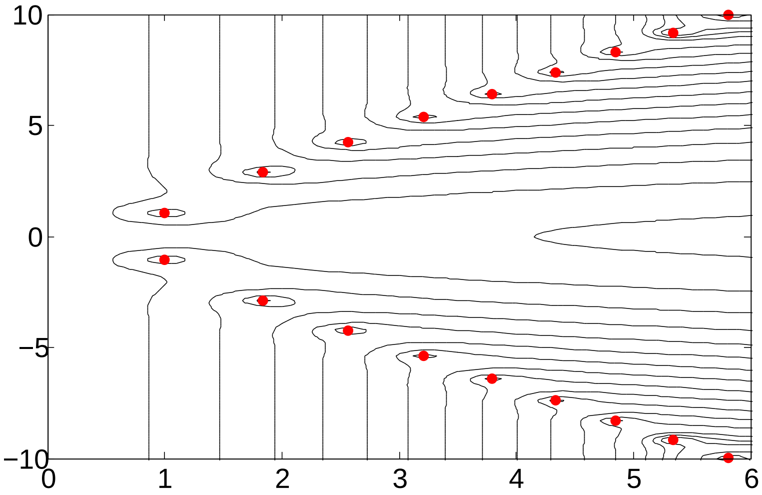

Figure: Numerically computed pseudospectrum in the complex plane of the complex Airy operator with the transmission boundary condition at the origin with $\kappa = 1$. The red points show the poles $\lambda_n^{\pm}(\kappa)$.

Basic properties of the Airy function

We recall that the Airy function is the unique solution of

$$(D_x^2 + x)u = 0\,,$$

on the line such that $u(x)$ tends to 0 as $x \to +\infty$ and $\mathrm{Ai}(0) = 1/\left(3^{\frac{2}{3}}\,\Gamma(\frac{2}{3})\right)$. This Airy function extends into an holomorphic function in $\mathbb{C}$.

Ai is positive decreasing on $\mathbb{R}_+$ but has an infinite number of zeros in $\mathbb{R}_-$. We denote by a_n ($n \in \mathbb{N}$) the decreasing sequence of zeros of Ai. Similarly we denote by a_n' the sequence of zeros of Ai'. Moreover

$$a_n \underset{n \to +\infty}{\sim} -\left(\frac{3\pi}{2}(n - 1/4)\right)^{2/3}\,, \tag{5}$$

and

$$a_n' \underset{n \to +\infty}{\sim} -\left(\frac{3\pi}{2}(n - 3/4)\right)^{2/3}\,. \tag{6}$$

$\mathrm{Ai}(e^{i\alpha}z)$ and $\mathrm{Ai}(e^{-i\alpha}z)$ (with $\alpha = 2\pi/3$) are two independent solutions of the differential equation

$$\left(-\frac{d^2}{dz^2} - iz\right) w(z) = 0\,.$$

Considering their Wronskian, one gets

$$e^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}z)\mathrm{Ai}(e^{i\alpha}z) - e^{i\alpha}\mathrm{Ai}'(e^{i\alpha}z)\mathrm{Ai}(e^{-i\alpha}z) = \frac{i}{2\pi}\,,\ \forall\ z \in \mathbb{C}\,. \quad (7)$$

Note the identity

$$\mathrm{Ai}(z) + e^{-i\alpha}\mathrm{Ai}(e^{-i\alpha}z) + e^{i\alpha}\mathrm{Ai}(e^{i\alpha}z) = 0\,,\ \forall\ z \in \mathbb{C}\,. \quad (8)$$

The Airy function and its derivative satisfy different asymptotic:
(i) For $|\arg z| < \pi$,

$$\mathrm{Ai}(z) = \frac{1}{2}\pi^{-\frac{1}{2}} z^{-1/4} \exp\left(-\frac{2}{3}z^{3/2}\right)\left(1+\mathcal{O}(|z|^{-\frac{3}{2}})\right), \quad (9)$$

$$\mathrm{Ai}'(z) = -\frac{1}{2}\pi^{-\frac{1}{2}}\, z^{1/4} \exp\left(-\frac{2}{3}z^{3/2}\right)\left(1+\mathcal{O}(|z|^{-\frac{3}{2}})\right). \quad (10)$$

(ii) For $|\arg z| < \frac{2}{3}\pi$,

$$\begin{aligned}\mathrm{Ai}(-z) &= \pi^{-\frac{1}{2}} z^{-1/4} \Bigg(\sin\left(\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right)\left(1+\mathcal{O}(|z|^{-\frac{3}{2}}\right) \\ &\quad -\frac{5}{72}\left(\frac{2}{3}z^{\frac{3}{2}}\right)^{-1}\cos\left(\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right)\left(1+\mathcal{O}(|z|^{-\frac{3}{2}}\right)\Bigg)\end{aligned} \quad (11)$$

$$\begin{aligned}\mathrm{Ai}'(-z) &= -\pi^{-\frac{1}{2}} z^{1/4} \Bigg(\cos(\frac{2}{3}z^{3/2}+\frac{\pi}{4})(1+\mathcal{O}(|z|^{-\frac{3}{2}}))\,, \\ &\quad +\frac{7}{72}\left(\frac{2}{3}z^{3/2}\right)^{-1}\sin(\frac{2}{3}z^{3/2}+\frac{\pi}{4})(1+\mathcal{O}(|z|^{-\frac{3}{2}}))\Bigg)\,.\end{aligned} \quad (12)$$

Analysis of the resolvent of $\mathcal{A}^+$ on the line for $\lambda > 0$

On the line $\mathbb{R}$, $\mathcal{A}^+$ is the closure of the operator $\mathcal{A}_0^+$ defined on $C_0^\infty(\mathbb{R})$ by $\mathcal{A}_0^+ = D_x^2 + ix$.
This is now standard. A detailed description of the properties of $\mathcal{A}^+$ can be found in my book in Cambridge University Press (2013).

One can give the asymptotic control of the resolvent $(\mathcal{A}^+ - \lambda)^{-1}$ as $\lambda \to +\infty$.

We successively discuss the control in $\mathcal{L}(L^2(\mathbb{R}))$ and in the Hilbert-Schmidt space $\mathcal{C}^2(L^2(\mathbb{R}))$.

Note that the norm of the resolvent $(\mathcal{A}^+ - \lambda)^{-1}$ depends only on the real part of λ.

Control in $\mathcal{L}(L^2(\mathbb{R}))$.

Here we follow an idea present in an old paper of I. Herbst, the book of Davies and used in Martinet's PHD (see also [26]).

Proposition

For all $\lambda > \lambda_0$,

$$\|(\mathcal{A}^+ - \lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \sqrt{2\pi}\ \lambda^{-\frac{1}{4}} \exp\left(\frac{4}{3}\lambda^{\frac{3}{2}}\right)(1 + o(1))\,. \tag{13}$$

Proof

The proof is obtained by considering $\mathcal{A}^+$ in the Fourier space, i.e.

$$\widehat{\mathcal{A}}^+ = \xi^2 + \frac{d}{d\xi}\,. \tag{14}$$

The associated semi-group $T_t := \exp(-\widehat{\mathcal{A}}^+ t)$ is given by

$$T_t u(\xi) = \exp\left(-\xi^2 t - \xi t^2 - \frac{t^3}{3}\right) u(\xi - t)\,, \quad \forall\, u \in \mathcal{S}(\mathbb{R})\,. \tag{15}$$

T_t is the composition of a multiplication by $\exp(-\xi^2 t - \xi t^2 - \frac{t^3}{3})$ and of a translation by t.

Computing $\sup_\xi \exp(-\xi^2 t - \xi t^2 - \frac{t^3}{3})$ leads to

$$\| T_t\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq \exp\left(-\frac{t^3}{12}\right) . \tag{16}$$

It is then easy to get an upper bound for the resolvent. For $\lambda > 0$, we have

$$\begin{aligned} \|(\mathcal{A}^+ - \lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} &= \|(\widehat{\mathcal{A}}^+ - \lambda)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}))} && (17)\\ &\leq \int_0^{+\infty} \exp(t\lambda)\| T_t\|_{\mathcal{L}(L^2(\mathbb{R}))} dt && (18)\\ &\leq \int_0^{+\infty} \exp\left(t\lambda - \frac{t^3}{12}\right) dt\,. && (19) \end{aligned}$$

Control in Hilbert-Schmidt norm

As previously, we use the Fourier representation and analyze $\widehat{\mathcal{A}}^+$. Note that

$$\|(\widehat{\mathcal{A}}^+ - \lambda)^{-1}\|_{HS}^2 = \|(\mathcal{A}^+ - \lambda)^{-1}\|_{HS}^2 \tag{20}$$

We have then an explicit description of the resolvent by

$$(\widehat{\mathcal{A}}^+ - \lambda)^{-1} u(\xi) = \int_{-\infty}^{\xi} u(\eta) \exp\left(\frac{1}{3}(\eta^3 - \xi^3) + \lambda(\xi - \eta)\right) d\eta\,.$$

Hence, we have to compute

$$\|(\widehat{\mathcal{A}}^+ - \lambda)^{-1}\|_{HS}^2 = \int\int_{\eta<\xi} \exp\left(\frac{2}{3}(\eta^3 - \xi^3) + 2\lambda(\xi - \eta)\right) d\eta d\xi\,.$$

Again, this can be analyzed after a scaling in the spirit of the Laplace method.

Analysis of the resolvent for the Dirichlet realization in the half-line.

It is not difficult to define the Dirichlet realization $\mathcal{A}^{\pm,D}$ of $D_x^2 \pm ix$ on $\mathbb{R}_+$ (the analysis on the negative semi-axis is similar). One can use for example the Lax Milgram theorem and take as form domain

$$V^D := \{u \in H_0^1(\mathbb{R}_+)\,,\, x^{\frac{1}{2}} u \in L^2_+\}\,.$$

It can also be shown that the domain is

$$\mathcal{D}^D := \{u \in V^D\,,\, u \in H^2_+\}\,.$$

This implies

Proposition

The resolvent $\mathcal{G}^{\pm,D}(\lambda) := (\mathcal{A}^{\pm,D} - \lambda)^{-1}$ is in the Schatten class C^p for any $p > \frac{3}{2}$ (see [16] for definition), where $\mathcal{A}^{\pm,D} = D_x^2 \pm ix$ and the superscript D refers to the Dirichlet case.

More precisely we provide the distribution kernel $\mathcal{G}^{-,D}(x,y\,;\lambda)$ of the resolvent for the complex Airy operator $D_x^2 - ix$ on the positive semi-axis with Dirichlet boundary condition at the origin. Matching the boundary conditions, one gets

$$\mathcal{G}^{-,D}(x,y\,;\lambda) = \begin{cases} 2\pi \frac{\mathrm{Ai}(e^{-i\alpha}w_y)}{\mathrm{Ai}(e^{-i\alpha}w_0)} \big[\mathrm{Ai}(e^{i\alpha}w_x)\mathrm{Ai}(e^{-i\alpha}w_0) \\ \qquad\qquad -\mathrm{Ai}(e^{-i\alpha}w_x)\mathrm{Ai}(e^{i\alpha}w_0)\big] \quad (0<x<y)\,, \\ 2\pi \frac{\mathrm{Ai}(e^{-i\alpha}w_x)}{\mathrm{Ai}(e^{-i\alpha}w_0)} \big[\mathrm{Ai}(e^{i\alpha}w_y)\mathrm{Ai}(e^{-i\alpha}w_0) \\ \qquad\qquad -\mathrm{Ai}(e^{-i\alpha}w_y)\mathrm{Ai}(e^{i\alpha}w_0)\big] \quad (x>y)\,, \end{cases} \tag{21}$$

where $\mathrm{Ai}(z)$ is the Airy function, $w_x = ix+\lambda$, and $\alpha = 2\pi/3$.
We have the decomposition

$$\mathcal{G}^{-,D}(x,y\,;\lambda) = \mathcal{G}_0^{-}(x,y\,;\lambda) + \mathcal{G}_1^{-,D}(x,y\,;\lambda), \tag{22}$$

where $\mathcal{G}_0^{-}(x,y\,;\lambda)$ is the resolvent for the Airy operator $D_x^2 - ix$ on the whole line,

$$\mathcal{G}_0^-(x,y\,;\lambda) = \begin{cases} 2\pi \mathrm{Ai}(e^{i\alpha} w_x)\mathrm{Ai}(e^{-i\alpha} w_y) & (x<y), \\ 2\pi \mathrm{Ai}(e^{-i\alpha} w_x)\mathrm{Ai}(e^{i\alpha} w_y) & (x>y), \end{cases} \tag{23}$$

and

$$\mathcal{G}_1^{-,D}(x,y\,;\lambda) = -2\pi \frac{\mathrm{Ai}(e^{i\alpha}\lambda)}{\mathrm{Ai}(e^{-i\alpha}\lambda)} \mathrm{Ai}\big(e^{-i\alpha}(ix+\lambda)\big)\, \mathrm{Ai}\big(e^{-i\alpha}(iy+\lambda)\big). \tag{24}$$

The resolvent is compact. The poles of the resolvent are determined by the zeros of $\mathrm{Ai}(e^{-i\alpha}\lambda)$, i.e., $\lambda_n = e^{i\alpha} a_n$, where the a_n are zeros of the Airy function: $\mathrm{Ai}(a_n) = 0$. The eigenvalues have multiplicity 1 (no Jordan block).

As a consequence of the analysis of the numerical range of the operator, we have

Proposition

$$||\mathcal{G}^{\pm,D}(\lambda)|| \leq \frac{1}{|\mathrm{Re}\,\lambda|}\,, \quad \textit{if}\ \mathrm{Re}\,\lambda < 0\,; \tag{25}$$

and

$$||\mathcal{G}^{\pm,D}(\lambda)|| \leq \frac{1}{|\mathrm{Im}\,\lambda|}\,, \quad \textit{if}\ \mp \mathrm{Im}\,\lambda > 0\,. \tag{26}$$

This proposition together with the Phragmen-Lindelöf principle (see Agmon [2] or Dunford-Schwartz [16])

Proposition

The space generated by the eigenfunctions of the Dirichlet realization $\mathcal{A}^{\pm,D}$ *of* $D_x^2 \pm ix$ *is dense in* L^2_+.

It is proven by R. Henry in [28] that there is no Riesz basis of eigenfunctions.

The Hilbert-Schmidt norm of the resolvent for $\lambda > 0$

At the boundary of the numerical range of the operator, it is interesting to analyze the behavior of the resolvent. Numerical computations lead to the observation that

$$\lim_{\lambda \to +\infty} ||\mathcal{G}^{\pm,D}(\lambda)||_{\mathcal{L}(L^2_+)} = 0 \,. \tag{27}$$

As a new result, we will prove

Proposition

When λ tends to $+\infty$, we have

$$||\mathcal{G}^{\pm,D}(\lambda)||_{HS} \approx \lambda^{-\frac{1}{4}} (\log \lambda)^{\frac{1}{2}} \,. \tag{28}$$

About the proof

The Hilbert-Schmidt norm of the resolvent can be written as

$$||\mathcal{G}^{-,D}||_{HS}^2 = \int_{\mathbb{R}_+^2} |\mathcal{G}^{-,D}(x,y\,;\lambda)|^2 dxdy = 8\pi^2 \int_0^{\infty} Q(x;\lambda)dx, \tag{29}$$

where

$$Q(x;\lambda) = \frac{|\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2}{|\mathrm{Ai}(e^{-i\alpha}\lambda)|^2} \times \times \int_0^x \left|\mathrm{Ai}(e^{i\alpha}(iy+\lambda))\mathrm{Ai}(e^{-i\alpha}\lambda) - \mathrm{Ai}(e^{-i\alpha}(iy+\lambda))\mathrm{Ai}(e^{i\alpha}\lambda)\right|^2 dy. \tag{30}$$

Using the identity (8), we observe that

$$\begin{aligned} &\mathrm{Ai}(e^{i\alpha}(iy+\lambda))\mathrm{Ai}(e^{-i\alpha}\lambda) - \mathrm{Ai}(e^{-i\alpha}(iy+\lambda))\mathrm{Ai}(e^{i\alpha}\lambda) \\ &\quad = e^{-i\alpha}\left(\mathrm{Ai}(e^{-i\alpha}(iy+\lambda))\mathrm{Ai}(\lambda) - \mathrm{Ai}(iy+\lambda)\mathrm{Ai}(e^{-i\alpha}\lambda)\right)\,. \end{aligned} \tag{31}$$

Hence we get

$$Q(x;\lambda) = |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2 \int_0^x \left|\mathrm{Ai}(e^{-i\alpha}(iy+\lambda))\frac{\mathrm{Ai}(\lambda)}{\mathrm{Ai}(e^{-i\alpha}\lambda)} - \mathrm{Ai}(iy+\lambda)\right|^2 dy\,. \tag{32}$$

More on Airy expansions

As a consequence of (9), we can write for large real λ

$$|\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))| = \frac{\exp\left(-\frac{2}{3}\lambda^{3/2}u(x/\lambda)\right)}{2\sqrt{\pi}(\lambda^2+x^2)^{1/8}}(1+\mathcal{O}(\lambda^{-\frac{3}{2}})), \tag{33}$$

where

$$\begin{aligned} u(s) &= -(1+s^2)^{3/4}\cos\left(\frac{3}{2}\tan^{-1}(s)\right) \\ &= \frac{\sqrt{\sqrt{1+s^2}+1}\,(\sqrt{1+s^2}-2)}{\sqrt{2}}. \end{aligned} \tag{34}$$

We note indeed that $|e^{-i\alpha}(ix+\lambda)| = \sqrt{x^2+\lambda^2} \geq \lambda \geq \lambda_0$ and that we have a control of the argument $\arg(e^{-i\alpha}(ix+\lambda)) \in [-\frac{2\pi}{3}, -\frac{\pi}{6}]$ which permits to apply (9).

Similarly, we obtain

$$|\mathrm{Ai}(ix+\lambda)| = \frac{\exp\left(\frac{2}{3}\lambda^{3/2}u(x/\lambda)\right)}{2\sqrt{\pi}(\lambda^2+x^2)^{1/8}}(1+\mathcal{O}(\lambda^{-\frac{3}{2}}))\,. \tag{35}$$

We note indeed that $|ix+\lambda| = \sqrt{x^2+\lambda^2}$ and that $\arg((ix+\lambda)) \in [0,+\frac{\pi}{2}]$ and one can then again apply (9). In particular the function $|\mathrm{Ai}(ix+\lambda)|$ grows super-exponentially as $x \to +\infty$.

Basic properties of u.

Note that

$$u'(s) = \frac{3}{2\sqrt{2}} \, \frac{s}{\sqrt{1+\sqrt{1+s^2}}} \geq 0 \qquad (s \geq 0), \tag{36}$$

and u has the following expansion at the origin

$$u(s) = -1 + \frac{3}{8} s^2 + \mathcal{O}(s^4)\,. \tag{37}$$

For large s, one has

$$u(s) \sim \frac{s^{3/2}}{\sqrt{2}}\,, \qquad u'(s) \sim \frac{3 s^{1/2}}{2\sqrt{2}}\,. \tag{38}$$

One concludes that the function u is monotonously increasing from -1 to infinity.

Upper bound

We start from the simple upper bound (for any $\epsilon > 0$)

$$Q(x,\lambda) \leq (1+\frac{1}{\epsilon})Q_1(x,\lambda) + (1+\epsilon)Q_2(x,\lambda)\,, \tag{39}$$

with

$$Q_1(x,\lambda) := |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2 \,\frac{|\mathrm{Ai}(\lambda)|^2}{|\mathrm{Ai}(e^{-i\alpha}\lambda)|^2} \int_0^x |\mathrm{Ai}(e^{-i\alpha}(iy+\lambda))|^2\, dy$$

and

$$Q_2(x,\lambda) := |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))|^2 \int_0^x |\mathrm{Ai}(iy+\lambda)|^2\, dy\,.$$

One can show that the integral of Q_1 is relatively small. It remains to estimate

$$\int_0^{+\infty} Q_2(x,\lambda)dx = \int_0^{+\infty} dx \int\limits_0^x |\mathrm{Ai}(e^{-i\alpha}(ix+\lambda))\mathrm{Ai}(iy+\lambda)|^2\, dy\,. \quad (40)$$

Using the estimates (33) and (35), we obtain

Lemma

There exist C *and* ϵ_0*, such that, for any* $\epsilon \in (0, \epsilon_0)$*, for* $\lambda > \epsilon^{-\frac{2}{3}}$*, the integral of* $Q_2(x; \lambda)$ *can be bounded as*

$$\frac{1}{2}(1 - C\epsilon)\, I(\lambda) \ \leq 8\pi^2 \int_0^{+\infty} Q_2(x, \lambda) dx \leq \frac{1}{2}(1 + C\epsilon)\, I(\lambda)\,, \tag{41}$$

where

$$I(\lambda) = \int_0^{\infty} dx\ \frac{\exp\left(-\frac{4}{3}\lambda^{3/2} u(x/\lambda)\right)}{(\lambda^2 + x^2)^{1/4}} \int_0^{x} dy\ \frac{\exp\left(\frac{4}{3}\lambda^{3/2} u(y/\lambda)\right)}{(\lambda^2 + y^2)^{1/4}}. \tag{42}$$

Control of $I(\lambda)$.

It remains to control $I(\lambda)$ as $\lambda \to +\infty$. Using a change of variables, we get

$$I(\lambda) = \lambda \int_0^{\infty} dx \, \frac{\exp\left(-\frac{4}{3}\lambda^{3/2}u(x)\right)}{(1+x^2)^{1/4}} \int_0^{x} dy \, \frac{\exp\left(\frac{4}{3}\lambda^{3/2}u(y)\right)}{(1+y^2)^{1/4}} \,. \tag{43}$$

Hence, introducing

$$t = \frac{4}{3}\lambda^{\frac{3}{2}} \,, \tag{44}$$

we reduce the analysis to $\hat{I}(t)$ defined for $t \geq t_0$ by

$$\hat{I}(t) := \int_0^{\infty} dx \, \frac{1}{(1+x^2)^{1/4}} \int_0^{x} dy \, \frac{\exp\big(t(u(y)-u(x))\big)}{(1+y^2)^{1/4}} \,, \tag{45}$$

with

$$I(\lambda) = \lambda \, \widehat{I}(t) \,. \tag{46}$$

The analysis is close to that of the asymptotic behavior of a Laplace integral.

Asymptotic upper bound of $\hat{I}(t)$.

Let us start by a heuristic discussion. The maximum of $u(y) - u(x)$ should be on $x = y$.
For $x - y$ small, we have $u(y) - u(x) \sim (y - x)u'(x)$.
This suggests a concentration near $x = y = 0$, whereas a contribution for large x is of smaller order.

The main term is

$$\widehat{I}_1(t,\epsilon) = \int_0^\epsilon dx \, \frac{1}{(1+x^2)^{1/4}} \int_0^x dy \, \frac{\exp\big(t(u(y)-u(x))\big)}{(1+y^2)^{1/4}} . \tag{47}$$

Its asymptotics is obtained using the asymptotics of

$$J_\epsilon(\sigma) := \int_0^\epsilon dx \int_0^x \exp\big(\sigma(y^2-x^2)\big)\, dy \, ,$$

which has now to be estimated for large σ.
Here appears the Dawson function (cf Abramowitz-Stegun [1], p. 295 and 319)

$$s \mapsto D(s) := \int_0^s \exp(y^2 - s^2)\, dy$$

and its asymptotics as $s \to +\infty$,

$$D(s) = \frac{1}{2s}\left(1 + \mathcal{O}(s^{-1})\right) . \tag{48}$$

Hence we have shown the existence of a constant $C > 0$ and of ϵ_0 such that if $t \geq C\epsilon^{-3}$ and $\epsilon \in (0, \epsilon_0)$

$$\widehat{l}_1(t,\epsilon) \leq \frac{2}{3}\frac{\log t}{t} + C(\epsilon\frac{\log t}{t} + \frac{1}{\epsilon}\frac{1}{t})\,. \tag{49}$$

Taking $\epsilon = (\log \lambda)^{-\frac{1}{2}}$, we obtain

Lemma

There exists $C > 0$ *and* λ_0 *such that for* $\lambda \geq \lambda_0$

$$||\mathcal{G}^{-,D}(\lambda)||_{HS}^2 \leq \frac{3}{8}\lambda^{-\frac{1}{2}} \log \lambda (1 + C\,(\log \lambda)^{-\frac{1}{2}})\,.$$

Lower bound

Once the upper bounds established, the proof of the lower bound is easy, using the simple lower bound (for any $\epsilon > 0$)

$$Q(x,\lambda) \geq -\frac{1}{\epsilon} Q_1(x,\lambda) + (1-\epsilon) Q_2(x,\lambda))\,. \tag{50}$$

Similar estimates to the upper bound give the proof of

Lemma

There exists $C > 0$ and λ_0 such that for $\lambda \geq \lambda_0$

$$||\mathcal{G}^{-,D}(\lambda)||_{HS}^2 \geq \frac{3}{8} \lambda^{-\frac{1}{2}} \log \lambda \,(1 - C\,(\log \lambda)^{-\frac{1}{2}})\,.$$

The complex Airy operator with a semi-permeable barrier: definition and properties

We consider the sesquilinear form a_ν defined for $u = (u_-, u_+)$ and $v = (v_-, v_+)$ by

$$\begin{aligned} a_\nu(u,v) &= \int_{-\infty}^{0} \Big(u_-'(x)\bar{v}_-'(x) + i\, x u_-(x)\bar{v}_-(x) + \nu\, u_-(x)\bar{v}_-(x) \Big)\, dx \\ &+ \int_{0}^{+\infty} \Big(u_+'(x)\bar{v}_+'(x) + i\, x u_+(x)\bar{v}_+(x) + \nu\, u_+(x)\bar{v}_+(x) \Big)\, dx \\ &+ \kappa\big(u_+(0) - u_-(0)\big)\overline{\big(v_+(0) - v_-(0)\big)}\,, \end{aligned} \tag{51}$$

where the form domain $\mathcal{V}$ is

$$\mathcal{V} := \left\{ u = (u_-, u_+) \in H^1_- \times H^1_+ : |x|^{\frac{1}{2}} u \in L^2_- \times L^2_+ \right\},$$

and $\nu \in \mathbb{R}$.

The space $\mathcal{V}$ is endowed with the Hilbertian norm

$$\|u\|_{\mathcal{V}} := \sqrt{\|u_-\|^2_{H^1_-} + \|u_+\|^2_{H^1_+} + \||x|^{1/2} u\|^2_{L^2}}\,.$$

We first observe that for any $\nu \geq 0$, the sesquilinear form a_ν is continuous on $\mathcal{V}$.

As the imaginary part of the potential $V(x) = ix$ changes sign, it is not straightforward to determine whether the sesquilinear form a_ν is coercive.

Due to the lack of coercivity, the standard version of the Lax-Milgram theorem does not apply. We shall instead use the following generalization introduced in Almog-Helffer [5].

Theorem

Let $\mathcal{V} \subset \mathcal{H}$ *be two Hilbert spaces s.t. that* $\mathcal{V}$ *is continuously embedded in* $\mathcal{H}$ *and dense in* $\mathcal{H}$. *Let* a *be a continuous sesquilinear form on* $\mathcal{V} \times \mathcal{V}$, *and* $\exists \alpha > 0$ *and two bounded linear operators* Φ_1 *and* Φ_2 *on* $\mathcal{V}$ *s.t.* $\forall u \in \mathcal{V}$,

$$\begin{cases} |a(u,u)| + |a(u,\Phi_1 u)| & \geq & \alpha \, \|u\|^2_{\mathcal{V}} \,, \\ |a(u,u)| + |a(\Phi_2 u,u)| & \geq & \alpha \, \|u\|^2_{\mathcal{V}} \,. \end{cases} \tag{52}$$

Assume further that Φ_1 *extends to a bounded linear operator on* $\mathcal{H}$. *Then* $\exists$ *a closed, densely-defined operator* S *on* $\mathcal{H}$ *with domain*

$$\mathcal{D}(S) = \{ u \in \mathcal{V} : v \mapsto a(u,v) \;\; \textit{can be extended continuously on} \;\; \mathcal{H} \},$$

s.t. $\forall u \in \mathcal{D}(S)$, $\forall v \in \mathcal{V}$,

$$a(u,v) = \langle Su, v \rangle_{\mathcal{H}} \,.$$

Moreover, from the characterization of the domain, we deduce the stronger

Proposition

There exists λ_0 *(*$\lambda_0 = 0$ *for* $\kappa > 0$*) such that* $(\mathcal{A}_1^+ - \lambda_0)^{-1}$ *belongs to the Schatten class* $\mathcal{C}^p$ *for any* $p > \frac{3}{2}$*.*

Note that if it is true for some λ_0 it is true for any λ in the resolvent set.

The following statement summarizes the previous discussion.

Proposition

The operator $\mathcal{A}_1^+$ *acting as*

$$u \mapsto \mathcal{A}_1^+ u = \left(-\frac{d^2}{dx^2}u_- + ixu_-, -\frac{d^2}{dx^2}u_+ + ixu_+\right)$$

on the domain

$$\mathcal{D}(\mathcal{A}_1^+) = \{u \in H_-^2 \times H_+^2 : xu \in L_-^2 \times L_+^2$$
$$\textit{and } u \textit{ satisfies transmission conditions } (3)\}$$

is a closed operator with compact resolvent.

$\exists \lambda > 0$ *s. t.* $\mathcal{A}_1^+ + \lambda$ *is maximal accretive.*

Remark

We have

$$\Gamma \mathcal{A}_1^+ = \mathcal{A}_1^- \,, \tag{53}$$

where Γ *denotes the complex conjugation:*

$$\Gamma(u_- \,,\, u_+) = (\bar{u}_- \,,\, \bar{u}_+)\,.$$

Remark (PT-Symmetry)

If (λ, u) *is an eigenpair, then* $(\bar{\lambda}, \bar{u}(-x))$ *is also an eigenpair.*

Integral kernel of the resolvent

Lengthy but elementary computations give:

$$\mathcal{G}^-(x,y\,;\lambda,\kappa) = \mathcal{G}_0^-(x,y\,;\lambda) + \mathcal{G}_1(x,y\,;\lambda,\kappa)\,, \tag{54}$$

where $\mathcal{G}_0^-(x,y\,;\lambda)$ is the distribution kernel of the resolvent of the operator $\mathcal{A}_0^* := -\frac{d^2}{dx^2} - ix$ on $\mathbb{R}$ and

$$\mathcal{G}_1(x,y\,;\lambda,\kappa) = \begin{cases} -4\pi^2 \frac{e^{2i\alpha}[\mathrm{Ai}'(e^{i\alpha}\lambda)]^2}{f(\lambda)+\kappa} \mathrm{Ai}(e^{-i\alpha}w_x)\mathrm{Ai}(e^{-i\alpha}w_y)\,, & \text{for } x > 0\,, \\ -2\pi \frac{f(\lambda)}{f(\lambda)+\kappa} \mathrm{Ai}(e^{i\alpha}w_x)\mathrm{Ai}(e^{-i\alpha}w_y)\,, & \text{for } x < 0\,, \end{cases} \tag{55}$$

for $y > 0$, and

$$\mathcal{G}_1(x,y\,;\lambda,\kappa) = \begin{cases} -2\pi \frac{f(\lambda)}{f(\lambda)+\kappa} \mathrm{Ai}(e^{-i\alpha}w_x)\mathrm{Ai}(e^{i\alpha}w_y)\,, & x > 0\,, \\ -4\pi^2 \frac{e^{-2i\alpha}[\mathrm{Ai}'(e^{-i\alpha}\lambda)]^2}{f(\lambda)+\kappa} \mathrm{Ai}(e^{i\alpha}w_x)\mathrm{Ai}(e^{i\alpha}w_y)\,, & x < 0\,, \end{cases} \tag{56}$$

for $y < 0$.

Hence the poles are determined by the equation

$$f(\lambda) = -\kappa\,, \tag{57}$$

with f defined by

$$f(\lambda) := 2\pi \mathrm{Ai}'(e^{-i\alpha}\lambda)\mathrm{Ai}'(e^{i\alpha}\lambda)\,. \tag{58}$$

Remark

For $\kappa = 0$, one recovers the conjugated pairs associated with the zeros a'_n of Ai'.

We have indeed

$$\lambda_n^+ = e^{i\alpha} a'_n\,, \quad \lambda_n^- = e^{-i\alpha} a'_n\,, \tag{59}$$

where a'_n is the n-th zero of Ai'.

We also know that the eigenvalues for the Neumann problem are simple. Hence by the local inversion theorem we get the existence of a solution close to each $\lambda_n^\pm$ for κ small enough (possibly depending on n) if we show that $f'(\lambda_n^\pm) \neq 0$. For λ_n^+, we have, using the Wronskian relation (7) and $\mathrm{Ai}'(e^{-i\alpha}\lambda_n^+) = 0$,

$$\begin{aligned} f'(\lambda_n^+) &= 2\pi\, e^{-i\alpha}\ \mathrm{Ai}''(e^{-i\alpha}\lambda_n^+)\mathrm{Ai}'(e^{i\alpha}\lambda_n^+) \\ &= 2\pi e^{-2i\alpha}\lambda_n^+\mathrm{Ai}(e^{-i\alpha}\lambda_n^+)\mathrm{Ai}'(e^{i\alpha}\lambda_n^+) \\ &= -i\lambda_n^+ \,. \end{aligned} \tag{60}$$

Similar computations hold for λ_n^-. We recall that

$$\lambda_n^+ = \overline{\lambda_n^-}\,.$$

Remark Very recently, we prove [AGH] that the eigenvalues are always simple.

Completion of the proof of the properties of the operator for the semi-permeable case

For the analysis of the resolvent, it is enough to compare the resolvent for some κ, with the resolvent for $\kappa = 0$, and to show that the asymptotic behavior as $\lambda \to +\infty$ is the same. For $\kappa = 0$, it is easy to see that we are reduced to the Neumann case on $\mathbb{R}^+$. We can also observe that the behavior of the resolvent are the same for Dirichlet and Neumann as $\lambda \to +\infty$.

Finally, we observe that we have the control of the resolvent on enough rays (the other rays being chosen outside of the numerical range) and the Phragmen-Lindelöf argument can now be used.

Then a general theorem (see the book of Agmon) gives us the completeness.

Applications to (2D)-problems

In higher dimension, an extension of the complex Airy operator is the differential operator that we call the Bloch-Torrey operator or simply the BT-operator:

$$-D\Delta + igx_1 \,,$$

where $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2$ is the Laplace operator in $\mathbb{R}^n$, and D and g are real parameters. More generally, we will study the spectral properties of some realizations of the differential operator

$$\mathcal{A}_h^{\#} = -h^2\Delta + i\,V(x)\,, \tag{61}$$

in an open set Ω, where h is a real parameter and $V(x)$ a real-valued potential with controlled behavior at ∞, and the superscript $\#$ distinguishes Dirichlet (D), Neumann (N), Robin (R), or transmission (T) conditions.

More precisely we discuss

1. the case of a bounded open set Ω with Dirichlet, Neumann or Robin boundary condition;
2. the case of a complement $\Omega := \complement\overline{\Omega}_-$ of a bounded set Ω_- with Dirichlet, Neumann or Robin boundary condition;
3. the case of two components $\Omega_- \cup \Omega_+$, with $\Omega_- \subset \overline{\Omega}_- \subset \Omega$ and $\Omega_+ = \Omega \backslash \overline{\Omega}_-$, with Ω bounded and transmission conditions at the interface between Ω_- and Ω_+;
4. the case of two components $\Omega_- \cup \complement\overline{\Omega}_-$, with Ω_- bounded and transmission conditions at the boundary;
5. the case of two unbounded components Ω_- and Ω_+ separated by a hypersurface and transmission conditions at the boundary.

The state u (in the first two items) or the pair (u_-, u_+) in the last items should satisfy some boundary or transmission condition at the interface. We consider the following situations:

- the Dirichlet condition: $u_{|\partial\Omega} = 0$;
- the Neumann condition: $\partial_\nu u_{|\partial\Omega} = 0$, where $\partial_\nu = \nu \cdot \nabla$, with ν being the outwards pointing normal;
- the Robin condition: $h^2 \partial_\nu u_{|\partial\Omega} = -\mathcal{K} u_{|\partial\Omega}$, where $\mathcal{K} \geq 0$ denotes the Robin parameter;
- the transmission condition:

$$h^2 \partial_\nu u_{+\,|\partial\Omega_-} = h^2 \partial_\nu u_{-\,|\partial\Omega_-} = \mathcal{K}(u_{+\,|\partial\Omega_-} - u_{-\,|\partial\Omega_-}) \,,$$

  where $\mathcal{K} \geq 0$ denotes the transmission parameter. In the last case, we should add a boundary condition at $\partial\Omega_+$ which can be Dirichlet or Neumann.

With Y. Almog and D. Grebenkov we have analyzed the spectral properties of the BT operator in dimension 2 or higher that are relevant for applications in superconductivity theory (Almog, Almog-Helffer-Pan, Almog-Helffer), in fluid dynamics (Martinet), in control theory (Beauchard-Helffer-Henry-Robbiano) and in diffusion magnetic resonance imaging (Grebenkov) . The main points to be solved or realized are:

- definition of the operator,
- construction of approximate eigenvalues in some asymptotic regimes,
- localization of quasimode states near certain boundary points,
- numerical simulations.

In particular, it is interesting to discuss the semiclassical asymptotics $h \to 0$, the large domain limit, the asymptotics when $g \to 0$ or $+\infty$, the asymptotics when the transmission or Robin parameter tends to 0. Some of these questions have been already analyzed by Y. Almog (see [3] (2008) and references therein for earlier contributions), R. Henry in his PHD (2013) (+ ArXiv paper 2014) and Almog-Henry (2015) but they were mainly devoted to the case of a Dirichlet realization in bounded domains in $\mathbb{R}^2$ or particular unbounded domains like $\mathbb{R}^2$ and $\mathbb{R}^2_+$, these two last cases playing an important role in the local analysis of the global problem.

We consider in Grebenkov-Helffer (2016) and In Almog-Grebenkov-Helffer (2017) $\mathcal{A}_h$ and the corresponding realizations in Ω are denoted by $\mathcal{A}_h^D$, $\mathcal{A}_h^N$, $\mathcal{A}_h^R$ and $\mathcal{A}_h^T$. These realizations can be properly under the condition that, when Ω is unbounded, there exists $C > 0$ such that

$$|\nabla V(x)| \leq C\sqrt{1 + V(x)^2}\,. \tag{62}$$

More details will be given in the talk by Y. Almog.

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