# Finite time blow-up in the two-dimensional harmonic map flow

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*Mostly Maximum Principle*
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**The harmonic map flow from $\mathbb{R}^2$ into $S^2$.**

$$u_t = \Delta u + |\nabla u|^2 u \quad \text{in} \quad \Omega \times (0, T) \qquad \text{(HMF)}$$

$$u = \varphi \quad \text{on } \partial\Omega \times (0, T)$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega$$

$u : \Omega \times [0, T) \to S^2$, $u_0 : \bar{\Omega} \to S^2$ smooth, $\varphi = u_0\big|_{\partial\Omega}$.
$\Omega$ smooth, bounded domain in $\mathbb{R}^2$ or entire space.

**Some characteristics of this flow:**

• The equation is the negative $L^2$-gradient flow for the Dirichlet energy $E(u) := \int_\Omega |\nabla u|^2 dx$. along smooth solutions $u(x,t)$:

$$\frac{d}{dt}E(u(\cdot,t)) \;=\; -\int_\Omega |u_t(\cdot,t)|^2 \;\leq 0 \;.$$

• The equation satisfies $|u(x,t)| = 1$ at all times if initial and boundary conditions do.

• The problem has blowing-up families of **energy invariant steady states** in entire space (entire harmonic maps).

Harmonic maps in $\mathbb{R}^2$ are solutions of

$$\boxed{\Delta u + |\nabla u|^2 u = 0, \quad |u| = 1 \text{ in } \mathbb{R}^2}$$

Example:

$$U_0(x) = \begin{pmatrix} \frac{2x}{1+|x|^2} \\ \frac{|x|^2-1}{1+|x|^2} \end{pmatrix}, \quad x \in \mathbb{R}^2.$$

The **1-corrotational harmonic maps**:

$$\boxed{U_{\lambda,x_0,Q}(x) = QU_0\left(\frac{x-x_0}{\lambda}\right)}$$

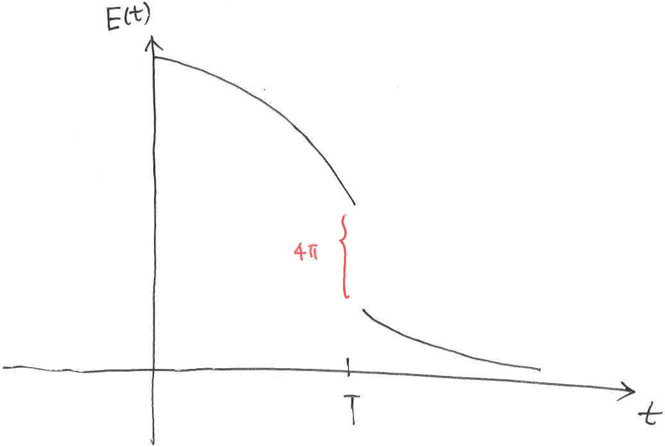
with $Q$ a linear orthogonal transformation of $\mathbb{R}^3$.

$$E_2(U_{\lambda,x_0,Q}) = E(U) \quad \text{for all} \quad \lambda, x_0.$$

• Local existence and uniqueness of a classical solution of (HMF): Eeels-Sampson (1966), Struwe (1984), K.C. Chang (1985)

• Struwe (1984): There exists a global $H^1$-weak solution of (HMF), where just for a finite number of points in space-time loss of regularity occurs.

• At those times jumps down in energy occur. This solution is unique within the class of weak solutions with degreasing energy, (Freire, 2002).




Energy jump of Struwe's Solution

E(t) is monotonely decreasing

If $T > 0$ designates the first instant at which smoothness is lost, we must have

$$\|\nabla u(\cdot, t)\|_\infty \to +\infty$$

Several works have clarified the possible blow-up profiles as $t \uparrow T$.

The following fact follows from results by Struwe 1984, Qing 1995 , Ding-Tian 1995 , Wang 1996, Lin-Wang 1998 and Qing-Tian 1997

Along a sequence $t_n \to T$ and points $q_1, \dots, q_k \in \Omega$, not necessarily distinct, $u(x, t_n)$ blows-up occurs at exactly those $k$ points in the form of *bubbling*. Precisely, we have

$$u(x, t_n) - u_*(x) - \sum_{i=1}^{k} \left[ U_i \left( \frac{x - q_i^n}{\lambda_i^n} \right) - U_i(\infty) \right] \to 0 \quad \text{in } H^1(\Omega)$$

where $u_* \in H^1(\Omega)$, $q_i^n \to q_i$, $0 < \lambda_i^n \to 0$, satisfy for $i \neq j$,

$$\frac{\lambda_i^n}{\lambda_j^n} + \frac{\lambda_j^n}{\lambda_i^n} + \frac{|q_i^n - q_j^n|^2}{\lambda_i^n \lambda_j^n} \to +\infty.$$

The $U_i$'s are entire, finite energy harmonic maps, namely solutions $U : \mathbb{R}^2 \to S^2$ of the equation

$$\Delta U + |\nabla U|^2 U = 0 \quad \text{n } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |\nabla U|^2 < +\infty.$$

After stereographic projection, $U$ lifts to a conformal smooth map in $S^2$, so that its value $U(\infty)$ is well-defined. It is known that $U$ is in correspondence with a complex rational function or its conjugate. Its energy corresponds to the absolute value of the degree of that map times the area of the unit sphere, and hence

$$\int_{\mathbb{R}^2} |\nabla U|^2 = 4\pi m, \quad m \in \mathbb{N},$$

In particular, $u(\cdot, t_n) \rightharpoonup u_*$ in $H^1(\Omega)$ and for some positive integers $m_i$, we have

$$|\nabla u(\cdot, t_n)|^2 \ \rightharpoonup \ |\nabla u_*|^2 + \sum_{i=1}^{k} 4\pi m_i \, \delta_{q_i}$$

$\delta_q$ denotes the Dirac mass at $q$.

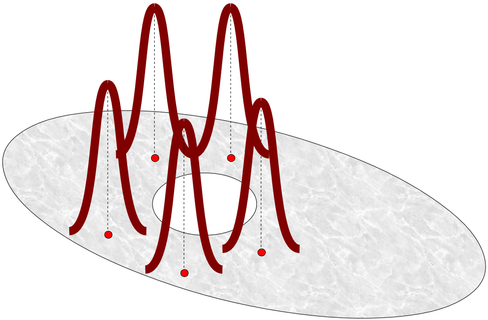
A *least energy* entire, non-trivial harmonic map is given by

$$U_0(x) \ = \ \frac{1}{1+|x|^2} \begin{pmatrix} 2x \\ |x|^2 - 1 \end{pmatrix}, \ x \in \mathbb{R}^2,$$

which satisfies

$$\int_{\mathbb{R}^2} |\nabla U_0|^2 = 4\pi, \quad U_0(\infty) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Expected shape of a **bubbling solution** as $t \uparrow T$



$$|\nabla u(x,t)|^2 \sim |\nabla u^*(x)|^2 + \sum_{j=1}^{k} \frac{1}{\lambda_j(t)^2} \left| \nabla U_i \left( \frac{x - q_j(t)}{\lambda_j(t)} \right) \right|^2$$

Very few examples are known of singularity formation phenomenon, all of them for single-point blow-up in radial *corrotational* classes.

When $\Omega$ is a disk or the entire space, a $1$-corrotational solution of (HMF) is one of the form

$$u(x,t) = \begin{pmatrix} e^{i\theta} \sin v(r,t) \\ \cos v(r,t) \end{pmatrix}, \quad x = re^{i\theta}.$$

(HMF) then reduces to

$$v_t = v_{rr} + \frac{v_r}{r} - \frac{\sin v \cos v}{r^2}$$

We observe that the function $w(r) = \pi - 2\arctan(r)$ is a steady state corresponding to to the harmonic map $U_0$:

$$U_0(x) = \begin{pmatrix} e^{i\theta} \sin w(r) \\ \cos w(r) \end{pmatrix}.$$

• Chang, Ding and Ye (1991) found the first example of a blow-up solution of Problem (HMF) (which was previously conjectured not to exist). It is a 1-corrotational solution in a disk with the blow-up profile $v(r,t) \sim w\left(\frac{r}{\lambda(t)}\right)$ or

$$u(x,t) \sim U_0\left(\frac{x}{\lambda(t)}\right).$$

and $0 < \lambda(t) \to 0$ as $t \to T$. No information on $\lambda(t)$

• Topping (2004) estimated the general blow-up rates as

$$\lambda_i = o(T-t)^{\frac{1}{2}}$$

(valid in more general targets), namely blow-up is of "type II": it does not occur at a self-similar rate.

• Angenent, Hulshof and Matano (2009) estimated the blow-up rate of 1-corrotational maps as $\lambda(t) = o(T-t)$.

• From formal analysis, van den Berg, Hulshof and King (2003) demonstrated that this rate for 1-corrotational maps should generically be given by

$$\lambda(t) \sim \kappa \frac{T-t}{|\log(T-t)|^2}$$

for some $\kappa > 0$.

• Raphael and Schweyer (2012) succeeded to rigorously construct a 1-corrotational solution with this blow-up rate in entire $\mathbb{R}^2$. Their proof provides the **stability** of the blow-up phenomenon within the radially symmetric class.

**A natural question:** The nonradial case: find nonradial solutions, single and multiple blow-up in entire space or bounded domains and analyze their stability.

**Our main result:** For any given finite set of points of $\Omega$ and suitable initial and boundary values, then a solution with a simultaneous blow-up at those points exists, with a profile resembling a translation, scaling and rotation of $U_0$ around each bubbling point. Single point blow-up is **codimension-1 stable**.

The functions

$$U_{\lambda,q,Q}(x) := QU_0\left(\frac{x-q}{\lambda}\right).$$

with $\lambda > 0$, $q \in \mathbb{R}^2$ and $Q$ an orthogonal matrix are least energy harmonic maps:

$$\int_{\mathbb{R}^2} |\nabla U_{\lambda,q,Q}|^2 = 4\pi.$$

For $\alpha \in \mathbb{R}$ we denote

$$Q_\alpha \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} e^{i\alpha}(y_1 + iy_2) \\ y_3 \end{bmatrix},$$

the $\alpha$-rotation around the third axis.

Let (HMF) with boundary condition $\varphi = U_0(\infty) = (0,0,1)$.

## Theorem (J. Dávila, M. del Pino, J. Wei)

*Let us fix points $q = (q_1, \dots, q_k) \in \Omega^k$. Given a sufficiently $T > 0$, there exists an initial condition $u_0$ such the solution $u_q(x,t)$ of (HMF) blows-up as $t \uparrow T$ in the form*

$$u_q(x,t) - u_*(x) - \sum_{j=1}^{k} Q_{\alpha_i^*}\left[\, U_0\left(\frac{x-q_i}{\lambda_i}\right) - U_0(\infty)\,\right] \;\to\; 0$$

*in the $H^1$ and uniform senses where $u_* \in H^1(\Omega) \cap C(\bar{\Omega})$,*

$$\lambda_i(t) = \frac{\kappa_i^*(T-t)}{|\log(T-t)|^2}.$$

$$|\nabla u_q(\cdot,t)|^2 \rightharpoonup |\nabla u_*|^2 + 4\pi \sum_{j=1}^{k} \delta_{q_j}$$

• Raphael and Schweyer (2013) proved the stability of their solution **within the 1-corrotational class**, namely perturbing slightly its initial condition in the associated radial equation the same phenomenon holds at a slightly different time.

• Formal and numerical evidence led van den Berg and Williams (2013) to conjecture that this radial bubbling *loses its stability* if special perturbations off the radially symmetric class are made. Our construction shows so at a linear level.

## Theorem (J. Dávila, M. del Pino, J. Wei)

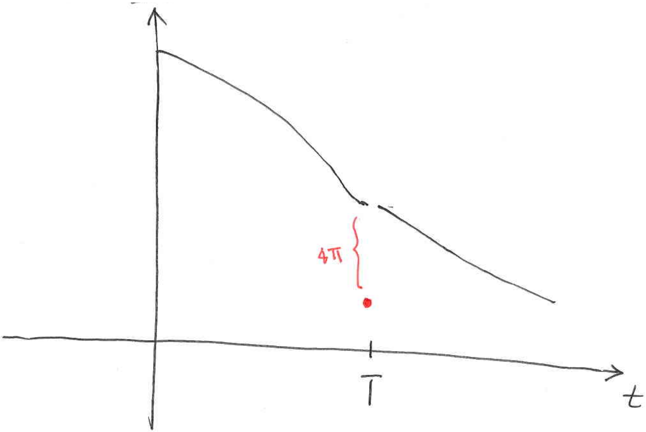
*For* $k = 1$ *there exists a manifold of initial data with codimension* $1$*, that contains* $u_q(x, 0)$*, which leads to the solution of (HMF) to blow-up at at exactly one point close to* $q$*, at a time close to* $T$*.*

**Continuation after blow-up?**

• Struwe defined a global $H^1$-weak solution of (HMF) by dropping the bubbles appearing at the blow-up time and then restarting the flow. This procedure modifies the topology of the image of $u(\cdot, t)$ across $T$.

• Topping (2002) built a continuation of Chang-Ding-Ye solution by **attaching a bubble with opposite orientation** after blow-up (this does not change topology and makes the energy values "continuous"). This procedure is called reverse bubbling. The reverse bubble is

$$\bar{U}_0(x) = \frac{1}{1+|x|^2}\begin{pmatrix} -2x \\ |x|^2 - 1 \end{pmatrix} = \begin{pmatrix} e^{i\theta}\sin\bar{w}(r) \\ \cos\bar{w}(r) \end{pmatrix}, \quad \bar{w}(r) = -w(r).$$




Reverse bubbling

E(t) is NOT monotonely decreasing

Theorem (J. Dávila, M. del Pino, J. Wei)

*The solution* $u_q$ *can be continued as an* $H^1$*-weak solution in* $\Omega \times (0, T+\delta)$, *with the property that* $u_q(x, T) = u_*(x)$

$$u_q(x,t) - u_*(x) - \sum_{j=1}^{k} Q_{\alpha_i^*}\left[\, \bar{U}_0\left(\frac{x-q_i}{\lambda_i}\right) - U_0(\infty)\,\right] \to 0 \quad \text{as } t \downarrow T,$$

*in the* $H^1$ *and uniform senses in* $\Omega$, *where*

$$\lambda_i(t) = \kappa_i^* \frac{t-T}{|\log(t-T)|^2} \quad \text{if } \; t > T.$$

It is reasonable to think that the blow-up behavior obtained is generic. Is it possible to have bubbles other than those induced by $U_0$ or $\bar{U}_0$, and or decomposition in several bubbles at the same point? Evidence seems to indicate the opposite:

• No blow-up is present in the higher corrotational class (Guan, Gustafson, Tsai, 2009).

• No *bubble trees* in finite time exist in the 1-corrotational class (Van der Hout 2002). In infinite time they do exist and their elements have been classified (Topping, 2004).

**Construction of a bubbling solution** $k = 1$

Given a $T > 0$, $q \in \Omega$, we want

$$S(u) := -u_t + \Delta u + |\nabla u|^2 u = 0 \quad \text{n } \Omega \times (0, T)$$

with

$$u(x,t) \approx U(x,t) := Q_{\alpha(t)} U_0 \left( \frac{x - x_0(t)}{\lambda(t)} \right)$$

The functions $\alpha(t), \ \lambda(t), \ x_0(t)$ are continuous with

$$\lambda(T) = 0, \quad x_0(T) = q.$$

We recall

$$U_0(y) = \begin{pmatrix} e^{i\theta} \sin w(\rho) \\ \cos w(\rho) \end{pmatrix}, \quad w(\rho) = \pi - 2\arctan(\rho), \quad y = \rho e^{i\theta},$$

We want to compute $S(U)$.

The vector fields

$$E_1(y) = \begin{pmatrix} -e^{i\theta} \cos w(\rho) \\ \sin w(\rho) \end{pmatrix}, \quad E_2(y) = \begin{pmatrix} ie^{i\theta} \\ 0 \end{pmatrix},$$

constitute an orthonormal basis of the tangent space to $S^2$ at the point $U_0(y)$.

$$S(U)(x,t) = Q_\alpha[\frac{\dot\lambda}{\lambda}\rho w_\rho \, E_1 \, + \, \dot\alpha \rho w_\rho \, E_2 \,] \, +$$
$$\frac{\dot x_{01}}{\lambda} \, w_\rho \, Q_\alpha[ \, \cos\theta \, E_1 + \sin\theta \, E_2] +$$
$$\frac{\dot x_{02}}{\lambda} \, w_\rho \, Q_\alpha[\sin\theta \, E_1 - \cos\theta \, E_2 \,].$$

For a small function $\varphi$, we compute

$$S(U+\varphi) = -\varphi_t + L_U(\varphi) + N_U(\varphi) + S(U).$$

$$L_U(\varphi) = \Delta\varphi + |\nabla U|^2\varphi + 2(\nabla U \nabla\varphi)U$$

$$N_U(\varphi) = |\nabla\varphi|^2 U + 2(\nabla U \nabla\varphi)\varphi + |\nabla\varphi|^2\varphi.$$

We need $|U+\varphi|^2 = 1$ or $2U\cdot\varphi + |\varphi|^2 = 0.$
If $\varphi$ is small, this approximately means

$$U\cdot\varphi = 0.$$

Neglecting quadratic terms, for small $\varphi$ we want:

$$\boxed{-\varphi_t + L_U(\varphi) + S(U) \approx 0, \quad \varphi\cdot U = 0.}$$

for a function $\varphi$ we write

$$\Pi_{U^\perp}\varphi := \varphi - (\varphi \cdot U)U.$$

We want to find a small function $\varphi^*$ such that

$$\boxed{-\partial_t \Pi_{U^\perp}\varphi^* + L_U(\Pi_{U^\perp}\varphi^*) + S(U) \approx 0.}$$

$\varphi^*$ will be made out of two pieces pieces $\varphi^* = \varphi^0 + Z^*$. For simplicity we fix

$$x_0 \equiv q, \quad \alpha \equiv 0.$$

**Step 1** Choice of $\varphi^0$ to concentrating the **outer** error. Far away from the concentration point the largest part of the error becomes

$$S(U)(x,t) \approx \mathcal{E}_0 = \frac{\dot\lambda}{\lambda}\rho w_\rho(\rho)\, E_1(y) \quad y = \frac{x - x_0}{\lambda} = \rho e^{i\theta}, \quad \rho = |y|.$$

Far away from $q$ we have

$$-\partial_t \Pi_{U^\perp} \varphi^0 + L_U[\Pi_{U^\perp} \varphi^0] + \mathcal{E}_0 \approx -\varphi_t + \Delta_x \varphi^0 - \frac{2}{r} \begin{bmatrix} e^{i\theta} \dot{\lambda} \\ 0 \end{bmatrix}.$$

so we require $\varphi^0 = \begin{bmatrix} e^{i\theta} \phi \\ 0 \end{bmatrix}$ where

$$\phi_t = \phi_{rr} + \frac{\phi_r}{r} - \frac{\phi}{r^2} - \frac{2\dot{\lambda}}{r} \; = \; 0.$$

We solve this equation with the aid of Duhamel's formula,

$$\phi = \phi_0[\dot{\lambda}](r,t) = -2 \int_0^t \dot{\lambda}(s) \frac{1 - e^{-\frac{r^2}{4(t-s)}}}{2r} \, ds.$$

The new error gets *concentrated* near $q$.

**Step 2**

We consider $\varphi^* = vp^0 + Z^*$ for a small smooth function $z^*(x,t) = z_1^*(x,t) + iz_2^*(x,t)$ which solves the heat equation,

$$\begin{aligned} z_t^* = \Delta z^*, \quad & \text{in } \Omega \times (0,T), \\ z(x,t) = z_0(x) \quad & \text{in } \partial\Omega \times (0,T), \\ z(x,0) = z_0(x) \quad & \text{in } \partial\Omega\ . \end{aligned}$$

On $z_0(x)$ we assume the following. For a point $q_0$ close to $q$,

$$\begin{aligned} \operatorname{div} z_0(q_0) = \partial_{x_1} z_{01}(q_0) + \partial_{x_2} z_{02}(q_0) \ & < 0 \\ \operatorname{curl} z_0(q_0) = \partial_{x_1} z_{02}(q_0) - \partial_{x_2} z_{01}(q_0) \ & = \ 0 \\ z_0(q_0) = 0, \quad Dz_0(q_0) \text{ non-singular.} & \end{aligned}$$

We write

$$Z^*(x,t) = \begin{bmatrix} z^*(x,t) \\ 0 \end{bmatrix} = \begin{bmatrix} z_1^* + iz_2^* \\ 0 \end{bmatrix}$$

and compute the linear error

$$-\partial_t \Pi_{U^\perp} Z^* + L_U(\Pi_{U^\perp} Z^*) -$$

$$\frac{1}{\lambda}\rho w_\rho^2 \left[\,\mathrm{div}\, z^* E_1 + \mathrm{curl}\, z^*\, E_2\right]$$

$$\frac{1}{\lambda}\rho w_\rho^2 \left[\,\mathrm{div}\, \bar{z}^* \cos 2\theta + \mathrm{curl}\, \bar{z}^* \sin 2\theta\,\right] E_1$$

$$\frac{1}{\lambda}\rho w_\rho^2 \left[\,\mathrm{div}\, \bar{z}^* \sin 2\theta - \mathrm{curl}\, \bar{z}^* \cos 2\theta\,\right] E_2$$

$$+\ O(\rho^{-2})$$

**Step 3** Finding $\varphi$ which improves the full error, namely that solves

$$-\partial_t(\Pi_{U^\perp}(\varphi^0 + Z^*) + \varphi) + L_U(\Pi_{U^\perp}(\varphi^0 + Z^*) + \varphi) + S(U) \approx$$

$$-\partial_t\varphi + L_U(\varphi) + \mathcal{E}_* = 0, \quad \varphi \cdot U = 0$$

where $\mathcal{E}_* = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$,

$$\mathcal{E}_1 = \left[ \lambda^{-2} \frac{4}{(1+\rho^2)^2} \left[ \phi_0[-2\dot\lambda] + \lambda\rho \,\mathrm{div}\, z^* \right] + \frac{2\lambda^{-1}\dot\lambda}{\rho(1+\rho^2)} \right] E_1$$

$$\mathcal{E}_2 = \frac{4\lambda^{-1}\rho}{(1+\rho^2)^2} \left\{ \left[ d_1 \cos 2\theta + d_2 \sin 2\theta \right] E_1 + \left[ d_1 \sin 2\theta - d_2 \cos 2\theta \right] E_2 \right\}$$

$$\mathcal{E}_3 = \frac{4\lambda^{-1}\rho}{(1+\rho^2)^2} \,\mathrm{curl}\, z^* \, E_2 + (U \cdot Z^*) \frac{2\lambda^{-1}\dot\lambda\rho}{1+\rho^2} E_1 + O(\rho^{-2})$$

We recall: $z^*(q,0) = 0$, $\mathrm{curl}\, z^*(q,0) = 0$, $\mathrm{div}\, z^*(q,0) < 0$.

In order to find $\varphi$ which cancels at main order $\mathcal{E}_1$ we consider the problem of finding $\varphi$ which decays away from the concentration point and satisfies

$$L_U(\varphi) + \mathcal{E}_1 = 0 \quad \varphi \cdot U = 0.$$

the following is a necessary (and sufficient!) condition We need the orthogonality condition

$$\int_{\mathbb{R}^2} \mathcal{E}_1 \cdot Z_{01} = 0$$

where $Z_{01} = \rho w_\rho E_1$ which satisfies $L_U[Z_{01}] = 0$. **This relation amounts to an equation for $\lambda(t)$.**

After some computation the equation for $\lambda(t)$ becomes approximately

$$\int_0^{t-\lambda^2} \frac{\dot\lambda(s)}{t-s}\, ds \;=\; 4\mathrm{div}\, z^*(q,t)\ .$$

Assuming that $\log \lambda \sim \log(T-t)$ the equation is well-approximated by

$$-\dot\lambda(t)\log(T-t) + \int_0^{t} \frac{\dot\lambda(s)}{T-s}\, ds \;+\; 4\mathrm{div}\, z^*(q,t) \;=0.$$

which is explicitly solved as

$$\dot\lambda(t) = -\frac{\kappa}{\log^2(T-t)}(1+o(1))$$

The value of $\kappa$ is precisely that for which

$$\kappa\int_0^{T} \frac{ds}{(T-s)\log^2(T-s)} \;=\; -4\mathrm{div}\, z^*(q,T).$$

Then if $T$ is small we get the approximation

$$\dot{\lambda}(t) \approx \dot{\lambda}_0(t) := \frac{4|\log T|}{\log^2(T-t)} \operatorname{div} z^*(q, T)$$

Since $\lambda$ decreases to zero as $t \to T^-$, this is where we need the assumption

$$\operatorname{div} z^*(q, T) < 0.$$

With this procedure we then get a true reduction of the total error by solving $L_U[\varphi] + \mathcal{E}_j = 0$, $j = 1, 2$.

At last we find a new approximation of the solution of the type

$$U_*(x,t) = U_0\left(\frac{x-q}{\lambda}\right) + \Pi_{U^\perp}[\phi_0[-2\dot{\lambda}] + Z^*(x,t)] + \varphi_*(x,t)$$

where $\varphi_*(x,t)$ is a decaying solution to

$$L_U[\varphi_*] = \mathcal{E}_*, \quad \varphi_* \cdot U = 0.$$

To solve the full problem we consider consider

$$\lambda(t) = \lambda_0(t) + \lambda_1(t), \quad \alpha(t) = 0 + \alpha_1(t), \quad x_0(t) = q_0 + x_1(t).$$

The true perturbations $\lambda_1, \alpha_1$ approximately solve linear equations of the type

$$\int_0^{t-\lambda_0^2} \frac{\dot{\lambda}_1(s)}{t-s}\, ds \;=\; p_1(t)$$

$$\int_0^{t-\lambda_0^2} \frac{\dot{\alpha}_1(s)\lambda_0(s)}{t-s}\, ds \;=\; p_2(t)$$

which are approximated by

$$-\dot{\lambda}_1(t)\log(T-t) + \int_0^t \frac{\dot{\lambda}_1(s)}{T-s}\, ds = p_1(t).$$

$$-\dot{\alpha}_1(t)\lambda_0\log(T-t) + \int_0^t \frac{\lambda_0\dot{\alpha}_1(s)}{T-s}\, ds = p_1(t).$$

and can be explicitly solved.

Thanks for your attention