Removable singularities and entire solutions of elliptic equations with absorption

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Overview

- Equations with absorption term
- Removable singularities: previous results
- Removable singularities: new results (V., t.a. on *Trans. AMS*)
- Entire solutions (Galise-V. 2011, Galise-Koike-Ley-V. 2016)
- Entire subsolutions (Capuzzo Dolcetta-Leoni-V. 2014 and 2016)

:

Equations with absorption term

• Look at non-negative solutions u of second-order elliptic equations

$$F(x, u, Du, D^2u) - |u|^{s-1}u = f(x)$$

where Du is the gradient of u and D^2u the Hessian matrix, $F(x, t, \xi, X)$ is nondecreasing in $X \in Sym(\mathbb{R}^n)$, Lipschitz continuous in $\xi \in \mathbb{R}^n$ and nonincreasing in $t \in \mathbb{R}$, $s > 1$.

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• Our assumptions on F will imply that, if u is an entire solution (defined in $\Omega = \mathbb{R}^n$) or a solution in a proper domain $\Omega \subset \mathbb{R}^n$ and $u \geq 0$ on $\partial\Omega$, and $f(x) \leq 0$, then $u \geq 0$ in Ω.

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• Consequently, in the case $f \leq 0$, the above equation becomes

$$F(x, u, Du, D^2u) - u^s = f(x)$$

and belongs to the more general class of equations

$$F[u] - g(u) = f(x)$$

with a superlinear *absorption term* $g(u) \geq 0$.

Removable singularities

• Let $E \subset \Omega$ be a closed set and u a solution of a PDE in $\Omega \backslash E$. Then E will be called a *removable singularity* if u can be continued as a solution in the whole Ω.

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(i) maximal growth of a solution near an isolated singularity;

(ii) maximal size of removable singularities.

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(i) maximal growth of a solution near an isolated singularity;

(ii) maximal size of removable singularities.

- Classical references:

Gilbarg - Serrin, *J. Analyse Math.* (1956)

Serrin, *Arch. Rational Mech. Anal.* (1964; 1965); *Acta Math.* (1965)

Removable singularities: classical results

- **Theorem** (isolated singularities) *Let $n > 2$ be an integer. Let $u(x)$ be a classical solution of the Laplace equation $\Delta u = 0$ in the punctured ball $B_r \backslash \{0\}$ of $\mathbb{R}^n$. If $u(x) = o(\mathcal{E}(x))$ as $x \to 0$, then $E = \{0\}$ is a removable singularity.*

Fundamental solution: $\mathcal{E}(x) = \frac{1}{|x|^{n-2}}$.

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- **Theorem** (non-isolated singularities) *Let $n > 2$ be an integer. Let $u(x)$ be a bounded classical solutions $u(x)$ of the Laplace equation $\Delta u = 0$ in $\Omega \backslash E$. If $E \subset \Omega$ is a compact set with Riesz capacity $C_{n-2}(E) = 0$, then E is a removable singularity.*

Removable singularities: uniformly elliptic operators

- **Pure second order**:

Labutin (*Viscosity Solutions of Differential Equations and Related Topics*, Ishii ed., Kyoto (2002), introduce a capacity "ad hoc" to characterize the size of removable singular sets.

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- **Theorem** (Amendola, Galise, V., *Differential Integral Equations*, 2013) *Let the exponent* $\alpha^* = (n-1)\frac{\lambda}{\Lambda} - 1 \geq 0$. *Assume:*

$$F = F(\xi, X) \text{ uniformly elliptic } (\lambda, \Lambda)$$
$$|F(\eta, X) - F(\xi, X)| \leq b|\eta - \xi| \quad \forall \xi, \eta \in \mathbb{R}^n.$$

Let $u(x)$ *be a bounded solution of equation*

$$F(Du, D^2u) = f(x) \quad \text{in} \quad \Omega \backslash E$$

with $f(x)$ *continuous in* Ω. *The singular set* E *is removable if:*

$$C_{\alpha^*}(E) = 0, \text{ for } b = 0; \; C_\alpha(E) = 0 \text{ with } \alpha \in (0, \alpha^*), \text{ for } b > 0\,.$$

Removable singularities: absorption terms

- **A well known theorem**

(Brezis-Veron, *Arch. Rat. Mech. Anal.*, 1980/81)

Let $n \geq 3$. *For The isolated singularities of solutions of equation*

$$\Delta u - |u|^{s-1} u = 0$$

are removable for $s \geq \frac{n}{n-2}$.

Already known by Loewner-Nirenberg (*Contribution to Analysis*, 1974) for $s \geq \frac{n+2}{n-2}$.

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NO CONDITION ON THE GROWTH OF THE SOLUTIONS

- **Generalized** by Labutin (*Arch. Ration. Mech. Anal.*, 2000) to fully nonlinear uniformly elliptic equations (λ, Λ)

$$F(D^2 u) - |u|^{s-1} u = 0$$

for $n > 1 + \frac{\Lambda}{\lambda}$ with $s \geq \dfrac{\lambda(n-1) + \Lambda}{\lambda(n-1) - \Lambda}$.

Relaxing ellipticity assumptions on the main term

- Upper Partial Sum of Hessian eigenvalues, picking the largest $p \leq n$ eigenvalues e_k (nondecreasing order) of $D^2 u$:

$$\mathcal{P}_p^+(D^2 u) = \sum_{k=n-p+1}^{n} e_k(D^2 u) = \sup_{W \in \mathcal{G}_p} \mathrm{Tr}|_W (D^2 u).$$

[$\mathcal{G}_p$:= Grassmanian of p-dim subspaces;
Tr:= trace operator]

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[$\mathcal{G}_p$:= Grassmanian of p-dim subspaces; Tr:= trace operator]

- Convention: F degenerate elliptic iff $X \leq Y \quad \Rightarrow \quad F(X) \leq F(Y)$

[$X \leq Y \Leftrightarrow Y - X$ positive semidefinite]

- Dual operator: Lower Partial Sum of Hessian eigenvalues

$$\mathcal{P}_p^-(X) = -\mathcal{P}_p^+(-X) = \inf_{W \in \mathcal{G}_p} \mathrm{Tr}|_W (X) = \sum_{k=1}^{p} e_k(D^2 u)$$

- Notice: $\mathcal{P}_p^+$ is subadditive, $\mathcal{P}_p^-$ superadditive.

Motivation

- Case $p = n$: $\mathcal{P}_n^{\pm}(D^2u) = \Delta u$ (uniformly elliptic).
- Remind: F uniformly elliptic iff

$$X \leq Y \quad \Rightarrow \quad \lambda Tr(Y - X) \leq F(Y) - F(X) \leq \Lambda Tr(Y - X)$$

[$0 < \lambda \leq \Lambda$ ellipticity constants]

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- Case $p < n$: $\mathcal{P}_p^{\pm}$ is not uniformly elliptic.

For instance, see below: $n = 2$; $p = 1$.

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \le \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Y \quad \Rightarrow \quad \inf_{X \le Y} \frac{e_2(Y) - e_2(X)}{Tr(Y - X)} = 0.$$

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- Geometric problems related to mean partial curvatures:

Sha (*Invent. Math.* 1987), Wu (*Indiana Univ. Math. J.* 1987)

- Related papers:

Harvey-Lawson (*Comm. Pure Appl. Math.* 2009, *Surv. Differ. Geom.* 2013, *Indiana Univ. Math. J.* 2014), Caffarelli-Li-Nirenberg (*Comm. Pure Appl. Math.* 2013)

Removability without absorption term

- **Theorem** (Caffarelli-Li-Nirenberg, 2013) *Let $2 \leq p \leq n$ be an integer. If $u(x)$ is a bounded solution of equation*

$$\mathcal{P}_p^-(D^2u) = f(x) \quad in \quad \Omega \backslash E$$

with $f(x)$ continuous in Ω, and $E \subset M$, a closed smooth manifold s.t. $dim(M) = p-2$, then E is a removable singularity.

- **Theorem** (Harvey-Lawson, 2014) *The same holds true under the capacitary assumption $C_{p-2}(E) = 0$.*

- **Remark**. The above results generalize what is known for the Laplace operator $\Delta u = \mathcal{P}_n^\pm(D^2u)$ to the partial Laplacians $\mathcal{P}_p^\pm(D^2u)$ with $p < n$ simply substituting p to the dimension n.

New results with absorption term

- **Theorem 1** (isolated singularities) *Let n and p be positive integers such that $3 \leq p \leq n$, and Ω be a domain of $\mathbb{R}^n$. For $x_0 \in \Omega$, set $\Omega_0 = \{x \in \Omega : x \neq x_0\}$. Suppose F is a continuous degenerate elliptic operator satisfying*

$$\mathcal{P}_p^-(Y - X) \leq F(Y) - F(X) \leq \mathcal{P}_p^+(Y - X)$$

and f is a continuous function in Ω.

For $s \geq \frac{p}{p-2}$, any continuous viscosity solution $u(x)$ of equation

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in Ω_0 can be extended to a solution in all Ω.

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• **Remark**. For $n = 2$ this returns the result of Brezis-Veron: no condition on the solution $u(x)$.

Comments and generalizations

- The partial Laplacians $\mathcal{P}_p^{\pm}$ fit the assumptions of Theorem 1.
- **Optimal exponent**. If $1 < s < \frac{p}{p-2}$, taking

$C^{s-1} = (p_s - 2)(p_s - p)$ with $p_s = \frac{2s}{s-1}$, we get a solution

$$u(x) = C|x|^{-\frac{2}{s-1}}$$

of equation

$$\mathcal{P}_p^+(D^2u) - |u|^{s-1}u = 0 \quad in \quad \dot{\mathbb{R}}^n$$

which cannot be continued across zero.

- **Generalizations** to more general absorption terms. The

conclusion of Theorem 1 continues to hold true for equations

$$F(D^2u) - g(u) = f(x),$$

where g is any continuous real function such that

$$\liminf_{u\to\pm\infty} \frac{g(u)}{|u|^{\frac{2}{p-2}}u} > 0\,.$$

Further generalizations

• Degenerate elliptic operators of Pucci type (Galise-V., Differential Integral Equations 2016):

$$\widetilde{\mathcal{P}}_p^-(X) = \lambda \sum_{i=1}^{p} e_i^+(X) - \Lambda \sum_{i=1}^{p} e_i^-(X) = \inf_{\substack{W \in \mathcal{G}_p \\ \lambda I_W \leq A_W \leq \Lambda I_W}} Tr(A_W X_W),$$

$$\widetilde{\mathcal{P}}_p^+(X) = \Lambda \sum_{i=n-p+1}^{n} e_i^+(X) - \lambda \sum_{i=n-p+1}^{n} e_i^-(X) = \sup_{\substack{W \in \mathcal{G}_p \\ \lambda I_W \leq A_W \leq \Lambda I_W}} Tr(A_W X_W).$$

• Case $p = n$: Pucci extremal operators $\widetilde{\mathcal{P}}_p^\pm = \mathcal{M}_{\lambda,\lambda}^\pm$.

• General case $p \leq n$, $\lambda \leq 1 \leq \Lambda$: $\widetilde{\mathcal{P}}_p^- \leq \mathcal{P}_p^- \leq \mathcal{P}_p^+ \leq \widetilde{\mathcal{P}}_p^+$.

• Highly degenerate elliptic Pucci maximal operator, the sum of positive eigenvalues of the Hessian matrix (Diaz 2012):

$$\mathcal{M}_{0,1}^+(D^2u) = e_1^+(D^2u) + \cdots + e_n^+(D^2u).$$

Non-isolated singularities

- **Theorem 2** *Let n and p be positive integers such that $3 \leq p \leq n$. Let also $k \in \mathbb{N}$ be such that $n - p + 2 \geq 0$. Let E be a closed set in $\mathbb{R}^n$ such that $E \subset \Omega \cap \Gamma$, where Γ is a smooth manifold in $\mathbb{R}^n$ of codimension $k \in (n - p + 2, n)$, and set $\Omega_E \equiv \Omega \backslash E$. Suppose F is a degenerate elliptic operator as in Theorem 1 with $f \in C(\Omega)$.*

For $s \geq \frac{p-(n-k)}{p-(n-k)-2}$, any continuous viscosity solution $u(x)$ of equation

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in Ω_E can be extended to a solution in all Ω.

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- **Remark 1**. In the limit case $k = n$ this returns the result of Theorem 1 for isolated singularities.

- **Remark 2**. As far as we know, this is new also when F is uniformly elliptic.

Sketch of the proof (isolated singularities)

- Fundamental solution of the equation $\mathcal{P}_p^+(D^2u)=0$:

$$\mathcal{E}_p(x)=|x|^{-(p-2)}.$$

- Upper bound for subsolutions $u(x)$ of equation $\mathcal{P}_p^+(D^2u)-|u|^{s-1}u=f(x)$ in B_R^* (via Osserman barrier functions):

$$u(x)\le A|x|^{-\frac{2}{s-1}}+\max_{|z-x|\le\frac{1}{2}|x|}\{f^-(z)\}^{\frac{1}{s}}\quad in\quad \dot{B}_{R/2}\,.$$

- If $\frac{2}{s-1}<p-2$, or $s>\frac{p}{p-2}$, then $u(x)$ goes to infinity less rapidly than $\mathcal{E}_p(x)$ around the origin and a comparison argument shows that $u(x)$ is bounded above in $\dot{B}_{R/2}$.

- Corresponding estimates from below hold for supersolutions so that solutions are bounded around the origin and we are done.

- Case $s=\frac{p}{p-2}$: $u^{\pm}(x)=o(\mathcal{E}_p(x))$ is proved by viscosity since a solution u of equation $\mathcal{P}_p^+(D^2u)-|u|^{s-1}u=f(x)$ in B_ρ provides a solution $w(y)=\rho^{p-2}u(x)$ of the same equation with $\rho^p f(x)$ instead of $f(x)$ via the transformation $y=y_0+\frac{x-x_0}{\rho}$ in B_1.

Existence of entire solutions

- *Entire solutions* are defined in the whole $\mathbb{R}^n$.
- If $u \in C(\overline{\Omega})$ is a viscosity solution of equation

$$F[u] - |u|^{s-1}u = f(x)$$

in a bounded domain Ω of $\mathbb{R}^n$, it can be plainly continued to an entire solutions if entire solutions exists and u is the restriction of one of this solutions to $\overline{\Omega}$.

- Existence of entire solutions have been obtained by:

Brezis (*Appl. Math. Optim.* 1984) for the Laplace operator;

Esteban - Felmer - Quaas (*Proc. Edinburgh Math. Soc.* 2010) for pure second order fully nonlinear uniformly elliptic operators;

Galise - V. (*Int. J. Differ. Equations* 2011) for the generalization to the dependence on x and on the gradient;

Galise - Koike - Ley - V. (*J. Math. Anal. Appl.* 2016) in the case of superlinear dependence on the gradient.

Uniqueness of entire solutions

- **Theorem 1** (Galise-V.) *The equation*

$$F(x, D^2u) + H(x, Du) - |u|^{s-1}u = f(x)$$

with $F(x, O) = 0$ *and* $H(x, 0) = 0$ *has a unique entire solution under the following assumptions:*

- *F uniformly elliptic* (λ, Λ)
- *H Lipschitz continuous in the gradient variable, uniformly with respect to* x
- *F satisfies* $C^{1,1}$*-estimates in the sense that for a solution* $u \in C^2(B_{r_0}) \cap C(\overline{B}_{r_0})$ *of the equation* $F(x, D^2u) = 0$ *we have the estimate*

$$\|u\|_{C^{1,1}(B_{r_0})} \leq C\|u\|_{L^\infty(\partial B_{r_0})}$$

for positive constants C *and* r_0*.*

- *For a suitable universal constant $\theta > 0$ (see Caffarelli, Annals)*

$$\sup_{0<r<r_0} \left(\fint_{B_r(x)} |\beta_F(x,y)|^n dy \right)^{\frac{1}{n}} \leq \theta$$

for every $x \in \mathbb{R}^n$ where

$$\beta_F(x,y) = \sup_{\substack{X \in \mathcal{S}^n \\ X \neq O}} \frac{|F(x,0,0,X) - F(y,0,0,X)|}{\|X\|}$$

- *f continuous*

- **Remark 1**. This yields uniqueness for the prototype equation

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u) \pm |Du| - |u|^{s-1}u = f(x)$$

- **Remark 2**. If $f \leq 0$ then $u \geq 0$.

Superlinear gradient terms

- **Theorem 2** (Galise-Koike-Ley-V.) *The equation*

$$F(x, D^2u) + H(x, Du) - |u|^{s-1}u = f(x)$$

with $F(x, O) = 0$ *and* $H(x, 0) = 0$ *has a unique entire solution under the following assumptions:*

- *F uniformly elliptic* (λ, Λ)*, continuous in* x *with a modulus* ω_R *(Crandall-Ishii-Lions) s.t. for in* $x, y \in B_R$

$$F(x, X) - F(y, Y) \leq \omega_R(|x-y| + \varepsilon^{-1}|x-y|^2)$$

whenever

$$-\frac{3}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}:$$

- *for* $m \in (1, 2]$ *and* $s > m$

$$|H(x, p+q) - H(y, p)| \leq \omega(|x-y|)(|p|^m + 1) + \gamma_1|q| + \gamma_m(|p|^{m-1} + |q|^{m-1})|q|$$

with a modulus ω *and constants* γ_1, γ_m.

- *F satisfies the homogeneity assumption*

$$F(x,\sigma X)=\sigma F(x,X) \quad \textit{for all } \sigma\in(0,1)$$

- *H satisfies the concavity type assumption*

$$\sigma H(x,\sigma^{-1}p)-H(x,p)\le(1-\sigma)(-\underline{c}|p|^m+A) \quad \textit{for } \sigma\in(\sigma_0,1)$$

with $\underline{c},A>0$ *and* $\sigma_0\in(0,1)$

- *f is continuous and*

$$\limsup_{|x|\to\infty}\frac{f^-(x)}{|x|^\rho}<\infty \quad \textit{for} \quad \rho<\begin{cases}\frac{m(s-1)}{(m-1)s} & \text{if } 1<m\le\frac{2s}{s+1}\\ \frac{2(s-m)}{s(m-1)} & \text{if } \frac{2s}{s+1}<m\,.\end{cases}$$

- **Remark.** This yields uniqueness for the prototype equation

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u)+c_1|Du|-c_m|Du|^m-|u|^{s-1}u=f(x),$$

when $c_m(x)$ is a bounded uniformly continuous function which satisfies $c_m(x)\ge\underline{c}>0$ (concave Hamiltonian).

- Motivation. A superlinear gradient term arise for the value functions u in stochastic control problems (Lasry-P.L.Lions, Math. Ann. 1989).

Continuation through $\mathbb{R}^n$

- Suppose F and f are defined for all $x \in \mathbb{R}^n$ and satisfy the assumptions of the previous theorems so that there exists a unique entire solution $\tilde{u}$.

If $u \in C(\overline{\Omega})$ is a solution of equation

$$F[u] - |u|^{s-1}u = f(x)$$

with $f \leq 0$ in a bounded domain $\Omega \subset \mathbb{R}^n$ and $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω by the maximum principle.

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• Looking back at the construction of $\tilde{u}$, the sequence of solutions u_k of the Dirichlet problems

$$\begin{cases} F[u] - |u|^{s-1}u = f(x) & \text{in } B_k \\ u = 0 & \text{on } \partial B_k \end{cases}$$

in the balls $B_k \supset \overline{\Omega}$ is non-decreasing and

$$\tilde{u}(x) = \lim_{k\to\infty} u_k(x) \quad \text{in } \mathbb{R}^n.$$

A necessary and sufficient condition

• In order u has an entire continuation, u has to be equal to the unique entire solution $\tilde{u}$ in $\overline{\Omega}$.
Therefore, letting $\varphi(x)$ the trace of $u(x)$ on $\partial\Omega$, a necessary condition in order u has an entire continuation is that

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• The above condition is also sufficient.
In fact, the entire solution $\tilde{u}$ is equal to $\lim_{k\to\infty} u_k$, by construction, and therefore equal to φ, by assumption, on $\partial\Omega$.
Thus both $\tilde{u}(x)$ and $u(x)$ are solution of the Dirichlet problem

$$\begin{cases} F[v] - |v|^{s-1}v = f(x) & \text{in } \Omega \\ v = \varphi & \text{on } \partial\Omega\,. \end{cases}$$

and again by comparison principle $\tilde{u} = u$ in Ω so that $\tilde{u}$ is actually an entire continuation of u.

Absorption terms and subsolutions

- Suppose now $f \geq 0$ and more generally a continuous non-negative g function on $[0, \infty)$, so that a non-negative solution of equation

$$F[u] - g(u) = f(x)$$

is in turn a subsolution of the associated homogeneous equation:

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• **Theorem** (Felmer-Quaas-Sirakov, J. Differential Equations 2013)
Let $g, h \in C[0, \infty)$ *be strictly increasing with* $g(0) = 0 = h(0)$ *and set*

$$G(t) = \int_0^t g(s)ds.$$

If at least one of conditions

$$\int_1^{+\infty} \frac{dt}{\sqrt{G(t)}} < \infty, \quad \int_1^{+\infty} \frac{dt}{h(t)} < \infty$$

is satisfied, then the differential inequality

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) - h(|Du|) - g(u) \geq 0$$

cannot have entire solutions. The same holds true for equation

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + h(|Du|) - g(u) \geq 0$$

if the following condition holds true:

$$\int_1^\infty \frac{ds}{K^{-1}(G(s))} < \infty,$$

being

$$K(s) = \int_0^s h(t)\,dt + 2ns^2.$$

- **Remark**. If $h \equiv 0$, this is the well known Keller-Osserman condition of non-existence: for instance, $g(t) >> t^{1+\varepsilon}$ with $\varepsilon > 0$, as $t \to \infty$. This can be weakened in the case of a negative gradient term while it has to be strenghtened in the case of positive sign.

If $h(t) = t^q$ with $q > 1$, then $K^{-1}(t) \approx t^{\frac{1}{q+1}}$, and the condition is satisfied for instance $g(t) \geq t^\alpha$ with $\alpha = \frac{q+1}{2} > 1$.

A slightly different absorption term

- Suppose now to consider the equation

$$F[u] - g(u) = f(x).$$

where $g(u) \geq 0$, so absorption independently of the sign of u.
Again supposing $f \geq 0$, a solution u is in turn a subsolution of the associated homogeneous equation:

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- This goes back to a well known result by Keller, Osserman 1957 in the case that $F[u] = \Delta u$: *if $f : \mathbb{R} \to \mathbb{R}$ is positive, continuous and nondecreasing, then the existence of entire subsolutions is equivalent to*

$$\int_1^{+\infty} \frac{dt}{\sqrt{G(t)}} = \infty.$$

Recent results

Theorem 1 *Let* $1 \leq p \leq n$ *and* $f : \mathbb{R} \to \mathbb{R}$ *be positive, continuous and nondecreasing. Let* F *be either uniformly elliptic or* $\mathcal{P}_p^+$*. Then the inequality*

$$F(D^2u) - g(u) \geq 0$$

has entire viscosity solutions if and only if f *satisfies the opposite Keller-Osserman condition:*

$$\int_1^\infty \frac{dt}{\sqrt{G(t)}} = \infty, \quad G(t) = \int_0^t g(s)ds.$$

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Theorem 2 *Assuming in addition f is strictly increasing, then*

$$\mathcal{M}_{0,1}^+(D^2u) \geq f(u)$$

has entire viscosity solutions if and only if f satisfies the opposite Keller-Osserman condition.

[Capuzzo Dolcetta - Leoni - V., Bull. Inst. Math. Acad. Sinica, 2014]

Subtracting a positive superlinear gradient term

Theorem 4 (Capuzzo Dolcetta - Leoni - V, Math. Ann. 2016)

Let $1 \leq p \leq n$, $0 < q \leq 2$ *and* g, k *be continuous non-negative nondecreasing functions, with* g *positive, strictly increasing and* k *such that*

$$\lim_{t \to +\infty} k(t) > 0.$$

Suppose F *uniformly elliptic or* $F = \mathcal{M}^+_{0,1}$. *There exist entire viscosity subsolutions of equation*

$$F(D^2u) - g(u) - k(u)\,|Du|^q \geq 0$$

if and only if

$$\int_1^\infty \frac{dt}{\sqrt{G(t)}} = \infty, \quad q \leq 1,\ \int_1^\infty \frac{dt}{(K^+(t))^{1/(2-q)}} = \infty\,, \qquad (-)$$

where $K^+(t) = \int_0^t k^+(s)ds$.

Adding a positive superlinear gradient term

Theorem 5 (Capuzzo Dolcetta - Leoni - V, Math. Ann. 2016)

Let $0 < q \leq 2$ and g, k be continuous nondecreasing functions, with g positive, strictly increasing, and $k \leq 0$. Suppose F uniformly elliptic or $F = \mathcal{M}^+_{0,1}$. There exists entire viscosity subsolutions of equation

$$F(D^2u) - g(u) + k(u)\,|Du|^q \geq 0$$

if and only if

$$\int_1^\infty \left(\int_0^t e^{-2\int_s^t \left(\frac{k^-(\tau)}{g(\tau)}\right)^{2/q} g(\tau)\,d\tau} g(s)\,ds \right)^{-1/2} dt = \infty\,.$$

If in addition $k \leq -\varepsilon < 0$, the above is equivalent to:

$$\int_1^\infty \frac{dt}{(tg(t))^{1/2}}\,dt \quad + \quad \varepsilon^{1/q} \int_1^\infty \frac{dt}{g(t)^{1/q}} \;=\; \infty\,. \qquad (-)$$

Bernstein-Nagumo condition

- We are investigating the case $q > 2$.
- Our results depend on the maximal solutions of an ODE

$$\varphi'' = h(x, \varphi, \varphi').$$

and is based on the fact that on the boundary of the maximal interval the solutions become unbounded together with their first derivatives.

- According to a well known result of Nagumo, this is true up to h has a quadratic growth in the derivative.
- For higher order growth there exist bounded maximal solutions with unbounded derivarive on the boundary of the maximal interval.
- As before, in the case of superquadratic growth in the gradient, we expect non-existence of entire solutions when substracting, but this requires a different technique.

Beyond Nagumo ($q > 2$)

- Assume f, g to be nondecreasing continuous positive functions with f strictly increasing.
- The maximal solutions of IVP

$$\begin{cases} \varphi''(r) + \frac{p-1}{r}\varphi'(r) = f(\varphi(r)) + g(\varphi(r))|D\varphi(r)|^q, \ r > 0 \\ \varphi(0) = t_0, \quad \varphi'(0) = 0\,, \end{cases}$$

defined in a finite interval $[0, R]$, are bounded even though $\varphi'(r) \to \infty$ as $r \to R^-$.

- Actually, φ is Hölder continuous with exponent $\alpha = \frac{q-2}{q-1}$.
- As a consequence, the subsolutions of equation

$$G(D^2u) = f(u) + g(u)|Du|^q,$$

which are on the other side Hölder continuous with the same exponent (Capuzzo Dolcetta-Leoni-Porretta, Trans. AMS 2010), cannot be defined throghout the whole space.