Minkowski Content and Exceptional Sets for Brownian Paths

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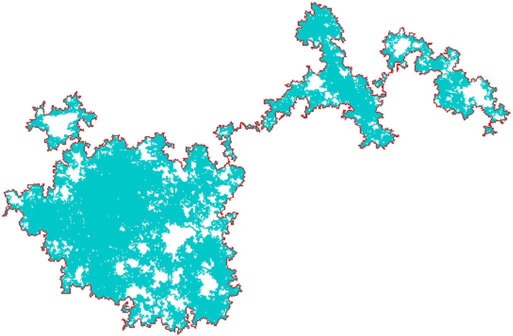
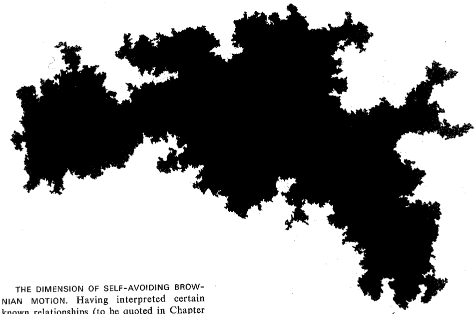
FRACTAL SUBSETS GENERATED BY BROWNIAN MOTION

B_t - standard Brownian motion in $\mathbb{R}^d$.

- The path $B[0,t]$.
- If $d=1$, the zero set $\{t : B_t = 0\}$
- The set of cut times $\{t : B[0,t) \cap B[t,1] = \emptyset\}$ or the corresponding set of cut points $\{B_t : B[0,t) \cap B[t,1] = \emptyset\}$.
  - No cut times for $d=1$; all cut times for $d \geq 4$.
  - Interesting for $d = 2,3$. (Burdzy)
- If $d=2$, the frontier or outer boundary of Brownian motion.
- Loop erasures.





THE DIMENSION OF SELF-AVOIDING BROWNIAN MOTION. Having interpreted certain known relationships (to be quoted in Chapter 36) as implying that a self-avoiding random walk is of dimension 4/3, I conjecture that the same is true of self-avoiding Brownian motion.

An empirical test of this conjecture provides an excellent opportunity to test also the length-area relation of Chapter 12. The plate is covered by increasingly tight square lattices, and we count the numbers of squares of side G intersected by a) the hull, standing for G-area, and b) its boundary, standing for G-length. Graphs relating G-length to G-area, using doubly logarithmic coordinates, were found to be remarkably straight, with a slope indistinguishable from $D/2=(4/3)/2=2/3$.

The resemblance between the curves in Plates 243 and 231, and their dimensions, is worth stressing.

NOTE. In Plate 243, the maximal open domains that B(t) does not visit are seen in gray. They can be viewed as tremas bounded by fractals, hence the loop is a net in the sense of Chapter 14.

◅ The question arises, of whether the loop is a gasket or a carpet from the viewpoint of the order of ramification. I conjectured that the latter is the case, meaning that Brown nets satisfy the Whyburn property, as described on p. 133. This conjecture has been confirmed in Kakutani & Tongling (unpublished). It follows that the Brown trail is a universal curve in the sense defined on page 144. ► ■

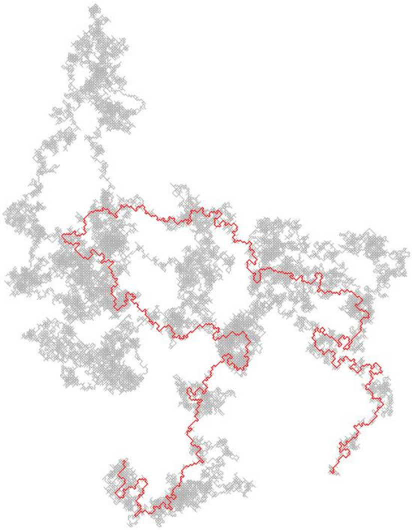


Figure: Loop-erased walk (F. Viklund)

Measuring the size of random fractal sets

- Minkowski or box dimension
- Hausdorff dimension
- Hausdorff measure (perhaps with a gauge function)
- Minkowski content

The goal of this talk is to discuss results about Minkowski content which is similar to local time.

Hausdorff measure

$$\mathcal{H}^\alpha_\epsilon(V) = \inf \sum [\mathrm{diam}\, U_j]^\alpha,$$

where the sum is over all covers of V with $\mathrm{diam}\, U_j \leq \epsilon$.

$$\mathcal{H}^\alpha(V) = \lim_{\epsilon \downarrow 0} \mathcal{H}^\alpha_\epsilon(V).$$

- Very nice properties — $\mathcal{H}^\alpha$ is a Borel measure.
- Can be refined by gauges

$$\mathcal{H}^\phi_\epsilon(V) = \inf \sum \phi(\mathrm{diam}\, U_j),$$

e.g., $\phi(r) = r^\alpha \, L(1/r)$ where L is slowly varying.

Hausdorff dimension

- $\dim_h(V) = \alpha$ if $\mathcal{H}^\beta(V) = \infty$ for $\beta < \alpha$ and $\mathcal{H}^\beta(V) = 0$ for $\beta > \alpha$.
- The value at $\mathcal{H}^\alpha(V)$ at $\alpha = \dim_h(V)$ can be 0, ∞, or something in between.
- Typically for random fractals $\mathcal{H}^\alpha(V) = 0$.
- The reason is that the infimum is taken over all covers of diameter $\leq \epsilon$. It is more natural, especially when considering limits from lattice models, to take infima over covers of diameter $= \epsilon$.
- For some fractals (Brownian path, local time), one can get a nontrivial value by correcting with a gauge function. This can be much harder for more complicated fractal sets arising from nonMarkov processes.

Minkowski content and dimension

- Let $V \subset \mathbb{R}^d$ be compact.
- $$\mathrm{Cont}_\alpha(V) = \lim_{\epsilon \downarrow 0} \epsilon^{\alpha - d} \, \mathrm{Vol}_d\{z : \mathrm{dist}(z, V) \le \epsilon\}.$$
- This is similar to finding optimal covers of V by balls of radius exactly ϵ.
- Typically this limit does not exists. We can define the upper content $\mathrm{Cont}_\alpha^+(V)$ by taking lim sup.
- (Upper) Minkowski or box dimension $\alpha = \dim_B(V)$ is defined by
$$\mathrm{Cont}_\beta^+(V) = \begin{cases} \infty, & \beta < \alpha \\ 0, & \beta > \alpha. \end{cases}$$
- $\dim_B(V) \ge \dim_h(V)$.

- Even though the Minkowski content is not defined for many sets, it is often the case that it is well defined (with probability one) for random fractals and gives a good "measure" on the set.
- It also gives quick definitions.
- For example if $Z_t = \{s \leq t : B_s = 0\}$ is the zero set for one-dimensional Brownian motion, then

$$L_t = \mathrm{Cont}_{1/2}(Z_t)$$

is well-defined and is (a constant times) the usual local time at 0 for the Brownian motion.

- Let B_t be a Brownian motion in $\mathbb{R}^d, d \geq 3$. Then

$$\mathrm{Cont}_2(B[0,t]) = c\,t,$$

for some easily computable constant c.
- Proved (although not stated like this) in, e.g., Le Gall's notes on Brownian motion.
- For $d = 2$ need a logarithmic correction essentially because the dimension of double points is the same as the dimension of the $B[0,t]$.
- If we were given the Brownian path but with the wrong parametrization, we could find the natural parametrization by using Minkowski content.

Upper bounds on dimension

- Suppose V is a random compact subset of $\mathbb{R}^d$.
- A weak one-point estimate

$$\mathbb{P}\{\mathrm{dist}(z, V) \leq \epsilon\} \lessapprox \epsilon^{\alpha}.$$

- Simple Markov inequality shows that with probability one $\dim_B(V) \leq d - \alpha$.
- When $\alpha > d$, then one shows that V is empty. (For example, the set of double points on the frontier of a Brownian loop).

Proving results about Hausdorff dimension

- Up-to-constants estimate

$$\mathbb{P}\{\mathrm{dist}(z, V) \leq \epsilon\} \asymp \epsilon^{\alpha}.$$

- Two-point estimate

$$\mathbb{P}\{\mathrm{dist}(z, V) \leq \epsilon, \mathrm{dist}(w, V) \leq \epsilon\} \leq c\, \epsilon^{2\alpha}\, |z - w|^{-\alpha}.$$

- Use estimate to put a find (with positive probability) a measure (Frostman measure) on V that is at least $(d - \alpha)$-dimensional.
- Generally defined as a subsequential limit — not necessary to show the limit exists.

Proving results about Minkowski content

- Need a strong one-point estimate.

$$\mathbb{P}\{\mathrm{dist}(z, V) \leq \epsilon\} = G(z)\, \epsilon^\alpha \,[1 + O(\epsilon^\beta)],$$

often proved by showing that

$$\mathbb{P}\{\mathrm{dist}(z, V) \leq e^{-(n+1)} \mid \mathrm{dist}(z, V) \leq e^{-n}\} = e^{-\alpha}\,[1+O(e^{-n\beta})].$$

- Independence of local behavior. Conditioned on

$$\{\mathrm{dist}(z, V) \leq e^{-n}, \mathrm{dist}(w, V) \leq e^{-n}\}$$

the events

$$\{\mathrm{dist}(z, V) \leq e^{-(n+1)}\}, \quad \{\mathrm{dist}(w, V) \leq e^{-(n+1)}\}$$

are almost independent.

Brownian frontier

- Mandelbrot saw a curve that looked like a SAW by viewing the outer boundary of random walk loop (Brownian bridge).
- This led to the conjecture that the dimension of the outer boundary of Brownian motion is $4/3$.
- For some of us seemed like a pretty wild conjecture!
- Burdzy noted that conjecture would imply something very unlikely — that one cannot tell the "inside" from the "outside" of the Brownian frontier if one only sees the frontier
- Several people (including me) tried (unsuccessfully!) to show that one could distinguish the inside from the outside.
- (Burdzy-L) The frontier of a Brownian bridge/loop is a Jordan curve. (Not true for a non-loop.)

- Let B^1, B^2 are independent Brownian motions and

$$T_n^j = \inf\{t : |B_t^j| = e^n\}, \quad \Gamma_n^j = B^j[T_0^j, T_n^j].$$

  Let A_n be the event that $\Gamma_n^1 \cup \Gamma_n^2$ does not disconnect the origin from infinity, $p_n = \mathbb{P}(A_n)$,
- There exists $\xi = \xi_2(2, 0)$ (disconnection exponent) such that $p_n \approx e^{-n\xi}$. This implies $\dim_B \leq 2 - \xi$.
- In fact, $\mathbb{P}(A_{n+1} \mid A_n) = e^{-\xi}\,[1 + O(\delta_n)]$, where δ_n summable. In particular $p_n \sim c\, e^{-n\xi}$ and $\dim_h = 2 - \xi$.
- Later work $\delta_n = O(e^{-\beta n})$.
- Exponent for random walk problem is the same. (L-Puckette)
- These techniques do not compute ξ although some estimates can be given.
- $\xi < 1$ and hence $\dim_h > 1$ (this last fact had been proved in a different way by Bishop, Jones, Pemantle, Peres)

- (L-Schramm-Werner) $\xi = 2/3$ and the dimension is $4/3$.
- In fact, the frontier is essentially a Schramm-Loewner evolution (SLE_κ) with parameter $8/3$.
- This is also the conjectured limit of self-avoiding walk. Mandelbrot's observations were correct!
- (Rohde-Schramm, Beffara) The Hausdorff dimension of SLE_κ paths is $1 + \frac{\kappa}{8}$.
- *SLE* paths were parameterized by capacity — this is singular with respect to the natural parametrization which would be scaling limit of counting measure.
- (L-Rezaei) If $\kappa < 8$, The $(1 + \frac{\kappa}{8})$-Minkowski content exists for SLE_κ paths and can be used to give the natural parametrization. (Earlier related work with Sheffield and Zhou.)
- In particular, the Brownian frontier can be parametrized by $\mathrm{Cont}_{4/3}$.

Open problem

- Let S_n be a simple random walk in $\mathbb{Z}^2$ conditioned so that $S_0 = S_{2N} = 0$.
- Let A be the path of the walk "filled in", that is, A is the smallest simply connected subset of $\mathbb{Z}^2$ containing all of the vertices in S_{2N}.
- View A as a simply connected domain D_A by replacing each vertex with the square of side length 1 centered at A.
- The boundary of D_A is a piecewise linear loop — parametrize this loop by length, giving a curve $\gamma_N(t), 0 \leq t \leq K$ where K is the number of edges.
- Conjecture: as $N \to \infty$, the distribution of the curve

$$\gamma^{(N)}(t) = N^{-4/3}\, \gamma_N(tN^{4/3}), \quad 0 \leq t \leq N^{-4/3}\, K$$

converges to the frontier of a Brownian bridge parametrized by (a constant times) the (4/3)-Minkowski content.

Really hard open problem

- Show that this also is the scaling limit for self-avoidng loops (polygons).
- Give each polygon of 2n steps measure $e^{-2n\beta_c}$ where β_c is critical, that is, the number of self-avoiding walks of length n grows like $e^{n\beta_c}$.
- The limiting measure should be the frontiers of the Brownian loop measure.

Cut points

- Let B^1, B^2 are independent Brownian motions and

$$T_n^j = \inf\{t : |B_t^j| = e^n\}, \qquad \Gamma_n^j = B^j[T_0^j, T_n^j].$$

  Let A_n be the event that $\Gamma_n^1 \cap \Gamma_n^2 = \emptyset$, $p_n = \mathbb{P}(A_n)$,
- There exists $\xi = \xi_d = \xi_d(1,1)$ (intersection exponent) such that $p_n \approx e^{-n\xi}$. This implies $\dim_B(\text{cutpoints}) \leq 2 - \xi$.
- Exponent for random walk problem is the same. (Burdzy-L)
- In fact, $\mathbb{P}(A_{n+1} \mid A_n) = e^{-\xi}\,[1 + O(\delta_n)]$, where δ_n summable. In particular $p_n \sim c\, e^{-n\xi}$ and $\dim_h(\text{cutpoints}) = 2 - \xi$.
- Later work $\delta_n = e^{-\beta n}$ (LSW, L-Vermesi)
- These techniques do not compute ξ although some estimates can be given.

$$\xi_d(1,2) = 4 - d, \qquad \frac{4-d}{2} < \xi_d < 4 - d.$$

- Numerics $\xi_3 \approx .58$. (Burdzy - L - Polaski) May never be determined exactly.
- (LSW) $\xi_2 = 5/4$ proved using SLE.

Theorem (in preparation, with N. Holden, X. Li, X, Sun)

- Consider the measure on Brownian paths starting at 0 ending at $x \neq 0$ in $\mathbb{R}^d$. (If $d = 3$, this has total mass $G(0, x)$ and has infinite mass in $d = 2$.)
- Consider the set of cut points on the path.
- Except for a set of paths of zero measure, the cut points have nontrivial $(2 - \xi)$-Minkowski content and this gives a function on the paths that is increasing only at the cut points.
- Important tool is the invariant measure on Brownian paths conditioned on a cut point. This is what is used to get

$$\mathbb{P}(A_{n+1} \mid A_n) = e^{-\xi} \, [1 + O(e^{-\beta n})].$$

Open problem

- Let $S_n, 0 \leq n \leq dN^2$ be a simple random walk in $\mathbb{Z}^d, d = 2, 3$.
- Consider the set of cut points on the walk and define

$$L_t = N^{\xi - 2} \, \#\{\text{cut points} \leq t \, N^2\}.$$

- Then the pair $(N^{-1} \, S_{tN^2}, L_t)$ converges to a Brownian motion with (a constant times) the Minkowski content of the cut points of the Brownian motion.
- One thing that is known is up-to-constant estimates for random walk,

$$\mathbb{P}\{S[0, N^2] \cap S[N^2 + 1, dN^2] = \emptyset\} \asymp N^{-\xi}.$$

(Similarly, up-to-contant estimates are known for the random walk frontier in $d = 2$.)

Why this problem arose

- Garban, Pete, and Schramm studied pivotal points for critical percolation on the triangular lattice for $d = 2$.
- They showed that counting measure, appropriately normalized, on the set of pivotal points had a scaling limit that is a measure on the whole scaling limit of percolation.
- The frontier of the scaling limit of percolation is the same as the frontier of Brownian motion. (Smirnov, LSW)
- Goal: to show that the measure they produced can be given by Minkowski content on the set of cut points of the Brownian motion.
- Here we are using the fact that cut points of the Brownian motion are cut points of the frontier.

One scaling limit that has been done

- Consider loop-erased random walk in $\mathbb{Z}^2$ parametrized by the number of steps.
- (LSW) If we ignore parametrization, the scaling limit is SLE_2.
- (L-Viklund) The scaling limit of the curves parametrized by the number of steps converges to SLE_2 parametrized by (a constant times) the Minkowski content.
- Proof requires both SLE estimates and a very strong estimate for the Green's function of the discrete loop-erased walk (Beneš-L-V).
  - Not just up-to-constant but asymptotic probabilities that are the same as for SLE and hence are conformally covariant.

Summary

- When parametrizing fractal sets arising from discrete limits, it is natural to use Minkowski content rather than versions of Hausdorff measure (when possible).
- There are (should be) many random fractals for which one can show the existence of the Minkowski content.
- Showing discrete limits may require deep understanding of the discrete object as well as the continuum.

THANK YOU

HAPPY 60th, CHRIS