Complete non-compact G_2–manifolds from asymptotically conical Calabi–Yau 3-folds

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joint with Mark Haskins and Johannes Nordström

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(M^n, g) complete Ricci-flat

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  - Minerbe (2011): assuming quadratic curvature decay and uniformly non-maximal volume growth the next possibility is $\mathrm{Vol}(B_r) = O(r^{n-1})$

- **Asymptotically locally conical (ALC)** manifolds: outside a compact set we have a circle fibration $M \setminus K \to \mathrm{C}(\Sigma)$ and the metric g is asymptotic to a Riemmanian submersion

$$g \sim g_{\mathrm{C}} + \theta^2$$

  - ALF gravitational instantons
  - **Higher dimensional ALC examples with holonomy G_2** and Spin_7
    2001: Brandhuber–Gomis–Gubser–Gukov, Cvetič–Gibbons–Lü–Pope

G_2–manifolds

M^7 orientable 7-manifold

- a **positive** 3-form φ:

$$\frac{1}{6}(u\lrcorner\varphi) \wedge (v\lrcorner\varphi) \wedge \varphi = g_\varphi(u,v)\,\mathrm{vol}_{g_\varphi}$$

- $\mathrm{Hol}(g_\varphi) \subseteq G_2 \iff d\varphi = 0 = d{*_\varphi}\varphi$ (**torsion-free G_2–structure**)
- Furthermore $\mathrm{Hol}(g_\varphi) = G_2 \iff (M, g_\varphi)$ carries no parallel 1-forms

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Dimensional reduction:

- $7 = 3 + 4$: G_2 and **hyperkähler** geometry

$$M^7 = \mathbb{R}^3 \times \mathrm{HK}^4, \qquad \varphi = dx_1 \wedge dx_2 \wedge dx_3 - dx_1 \wedge \omega_1 - dx_2 \wedge \omega_2 - dx_3 \wedge \omega_3$$

- $7 = 1 + 6$: G_2 and **Calabi–Yau** geometry

$$M^7 = \mathbb{R} \times \mathrm{CY}^3, \qquad \varphi = dx \wedge \omega + \mathrm{Re}\,\Omega$$

Main result

Theorem (F.–Haskins–Nordström, 2017)

Let $(B, g_0, \omega_0, \Omega_0)$ be an **asymptotically conical Calabi–Yau 3-fold** asymptotic to a Calabi–Yau cone $(\mathrm{C}, g_{\mathrm{C}})$ and let $M \to B$ be a **principal circle bundle**.

Assume that $c_1(M) \neq 0$ but $c_1(M) \cup [\omega_0] = 0$.

Then for every $\epsilon > 0$ sufficiently small there exists an **$\mathbf{S}^1$–invariant $\mathbf{G}_2$–holonomy metric g_ϵ** on M with the following properties.

- (M, g_ϵ) is an ALC manifold: as $r \to \infty$, $g_\epsilon = g_{\mathrm{C}} + \epsilon^2 \theta_\infty^2 + O(r^{-\nu})$.
- **(M, g_ϵ) collapses to (B, g_0) with bounded curvature as $\epsilon \to 0$**: $g_\epsilon \sim_{C^{k,\alpha}} g_0 + \epsilon^2 \theta^2$ as $\epsilon \to 0$.

Main result: comments

- Only 4 non-trivial examples of simply connected complete non-compact G_2–manifolds are currently known:
  - three asymptotically conical examples due to Bryant–Salamon (1989);
  - an explicit example due to Brandhuber–Gomis–Gubser–Gukov (2001) moving in a 1-parameter family whose existence was rigorously established by Bogoyavlenskaya (2013).

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- Non-compact complete examples of manifolds with special holonomy that collapse with globally bounded curvature are a **new higher-dimensional phenomenon**: the only hyperkähler 4-manifold with a tri-holomorphic circle action without fixed points is $\mathbb{R}^3 \times S^1$.

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  We produce infinitely many new examples.
- Non-compact complete examples of manifolds with special holonomy that collapse with globally bounded curvature are a **new higher-dimensional phenomenon**: the only hyperkähler 4-manifold with a tri-holomorphic circle action without fixed points is $\mathbb{R}^3 \times S^1$.
- **Connections to physics**: Type IIA String theory compactified on AC CY 3-fold (B, ω_0, Ω_0) with Ramond–Ramond 2-form flux $d\theta$ satisfying $[d\theta] \cup [\omega_0] = 0$ and no D6 branes nor $O6^-$ planes as the weak-coupling limit of M theory compactified on an ALC G_2–manifold.

The Gibbons–Hawking Ansatz

Recall the **Gibbons–Hawking Ansatz** (1978): local form of hyperkähler metrics in dimension 4 with a triholomorphic circle action

- U open subset of $\mathbb{R}^3$
- h positive function on U
- $M \to U$ a principal U(1)–bundle with a connection θ

$$g = h\, g_{\mathbb{R}^3} + h^{-1}\theta^2 \text{ is a hyperkähler metric on } M$$

$$\Updownarrow$$

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Goal: a higher-dimensional analogue for G_2–manifolds

Cvetič–Gibbons–Lü–Pope (2002), Kaste–Minasian–Petrini–Tomasiello (2003), Apostolov–Salamon (2004)

The Apostolov–Salamon equations

- $M^7 \to B^6$ a principal circle bundle with connection θ
- $h : B \to \mathbb{R}^+$
- (ω, Ω) an SU(3)–structure on B

$$\varphi = \theta \wedge \omega + h^{\frac{3}{4}} \operatorname{Re}\Omega, \qquad *_\varphi \varphi = -\theta \wedge h^{\frac{1}{4}} \operatorname{Im}\Omega + \tfrac{1}{2} h\,\omega^2,$$
$$g_\varphi = \sqrt{h}\, g_{\omega,\Omega} + h^{-1}\theta^2$$

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Torsion-free G_2–structure on M if and only if

- $\left(\frac{4}{3} h^{\frac{3}{4}}, \theta\right)$ satisfies the **Calabi–Yau monopole equations**

$$\tfrac{1}{2} dh \wedge \omega^2 = h^{\frac{1}{4}} d\theta \wedge \operatorname{Im}\Omega, \qquad d\theta \wedge \omega^2 = 0$$

- the SU(3)–structure (ω, Ω) has **constrained torsion**

$$d\omega = 0, \qquad d\left(h^{\frac{3}{4}} \operatorname{Re}\Omega\right) + d\theta \wedge \omega = 0, \qquad d\left(h^{\frac{1}{4}} \operatorname{Im}\Omega\right) = 0$$

Adiabatic limit of the AS equations

Introduce a small parameter $\epsilon > 0$:

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- **Formal limit as $\epsilon \to 0$**: $h_0 \equiv 1$ and (ω_0, Ω_0) is a **CY structure** on B.
- **Linearisation over the collapsed limit**:
  - **Calabi–Yau monopole**

    $$\tfrac{1}{2} dh \wedge \omega_0^2 = d\theta \wedge \mathrm{Im}\,\Omega_0, \qquad d\theta \wedge \omega_0^2 = 0$$

  - **infinitesimal deformation of the SU(3)–structure**

    $$d\dot{\omega} = 0, \qquad d\mathrm{Re}\,\dot{\Omega} + \tfrac{3}{4} dh \wedge \mathrm{Re}\Omega_0 + d\theta \wedge \omega_0 = 0, \qquad d\mathrm{Im}\,\dot{\Omega} + \tfrac{1}{4} dh \wedge \mathrm{Im}\Omega_0 = 0$$

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  - $h \equiv 0$ and θ is a Hermitian Yang–Mills (HYM) connection
  - (h, θ) has singularities (e.g. Dirac-type singularities along a special Lagrangian submanifold $L \subset B$)

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- **Proposition** Let $M \to B$ be a principal circle bundle over an irreducible AC Calabi–Yau 3-fold (B, ω_0, Ω_0). Then M carries a HYM connection θ.
  - By Hodge theory on AC manifolds we can represent every cohomology class in $H^2(B)$ with a unique closed and coclosed form of rate $O(r^{-2})$
  - There are no decaying harmonic functions and 1-forms on $B \implies$ the closed and coclosed representative of $c_1(M)$ is a primitive $(1,1)$–form

Solution of the AS equations at first order in ϵ

- $M \to B$ principal U(1)–bundle with HYM connection θ
- We look for an **infinitesimal deformation $(\dot{\omega}, \dot{\Omega})$ of the SU(3)–structure** such that

$$d\dot{\omega} = 0, \qquad d\mathrm{Re}\,\dot{\Omega} + d\theta \wedge \omega_0 = 0, \qquad d\mathrm{Im}\,\dot{\Omega} = 0$$

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- **Necessary and sufficient condition**

$$d\theta \perp_{L^2} L^2\mathcal{H}^2(B) \simeq H^2_c(B) \iff c_1(M) \cup [\omega_0] = 0 \in H^4(B) \simeq H^2_c(B)^*$$

Solving the AS equations for small ϵ

- (B, ω_0, Ω_0) AC Calabi–Yau manifold
- principal U(1)–bundle $M \to B$ with $c_1(M) \neq 0$ & $c_1(M) \cup [\omega_0] = 0$

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- Solution

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of the **linearised AS equations**

$\rightsquigarrow$ closed ALC S^1–invariant G_2–structure on M with torsion $O(\epsilon^2)$

$$\varphi_\epsilon^{(1)} = \epsilon\, \theta \wedge \omega_0 + \mathrm{Re}\, \Omega_0 - \epsilon *\rho$$

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- Construct formal solution of the non-linear AS equations as a formal **power series** in ϵ
- Prove the series has a **positive radius of convergence** (in weighted Hölder spaces)

The torsion of SU(3)–structures on 6-manifolds

- If (ω, Ω) is an SU(3)–structure then there exist $w_1, \hat{w}_1 \in \Omega^0$, $w_4, w_5 \in \Omega^1$, $w_2, \hat{w}_2 \in \Omega^4_8$ and $w_3 \in \Omega^3_{12}$ such that

$$d\omega = 3w_1 \,\mathrm{Re}\,\Omega + 3\hat{w}_1 \,\mathrm{Im}\,\Omega + w_3 + w_4 \wedge \omega,$$

$$d\mathrm{Re}\,\Omega = -2\hat{w}_1\,\omega^2 + w_5 \wedge \mathrm{Re}\,\Omega + w_2,$$

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- Introduce **free parameters** $f, g \in \Omega^0$ and $X \in \Omega^1$ $\rightsquigarrow$ **extended AS eqs**

$$\tfrac{1}{2} dh \wedge \omega^2 = h^{\frac{1}{4}} d\theta \wedge \operatorname{Im}\Omega, \qquad d\theta \wedge \omega^2 = 0,$$
$$d\omega = 0, \qquad d\left(h^{\frac{3}{4}} \operatorname{Re}\Omega\right) + d\theta \wedge \omega = d{*}d(f\,\omega),$$
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  - Need to use that there are no decaying elements in the kernel of

$$\pi_1\left(d^* d(f\omega)\right) \longleftrightarrow \triangle f \qquad \pi_{1\oplus 6}\left(d^* d(g\omega + X \lrcorner \mathrm{Re}\,\Omega)\right) \longleftrightarrow \triangle g,\, dd^* X + \tfrac{2}{3} d^* dX$$

The linearised AS equations

- The **extended linearised operator**

$$\mathcal{L} : 3\,\Omega^0 \oplus 2\,\Omega^1 \oplus \Omega^3 \longrightarrow 2\,\Omega^0 \oplus \Omega^1 \oplus 2\,\Omega^4_{exact}$$

$$\tfrac{1}{2} dh \wedge \omega_0^2 - d\gamma \wedge \mathrm{Im}\,\Omega_0, \qquad d\gamma \wedge \omega_0^2, \qquad d^*\gamma$$

$$d\left(\rho + \tfrac{3}{4} h\,\mathrm{Re}\,\Omega_0 + \gamma \wedge \omega_0\right) + d{*}d(f\,\omega)$$

$$d\left(\hat{\rho} + \tfrac{1}{4} h\,\mathrm{Im}\,\Omega_0\right) + d{*}d\left(g\,\omega + X \lrcorner \mathrm{Re}\,\Omega\right)$$

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- The **extended linearised operator**

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- Use the Dirac operator to derive **"normal forms" for exact 4-forms** and thereby relate the remaining two equations to $(d + d^*)\rho$
- The extended linearised operator $\mathcal{L}$ is **surjective** and has a **bounded** right inverse in appropriate weighted Hölder spaces
- **Existence** and **convergence** of power series solutions to the AS eqs

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Consider the isolated hypersurface singularity $X_p \subset \mathbb{C}^4$ defined by

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- circle bundle $M \to B$ has $b_2(M) = p - 1$ and $b_3(M) = p$

$\rightsquigarrow$ **infinitely many new simply connected complete G_2–manifolds** and **families of complete non-compact G_2–metrics of arbitrarily high dimension**