Variational integrator methods for fluid-structure interactions

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Connections in Geometric Numerical Integration and Structure-Preserving Discretization

Joint work with Francois Gay-Balmaz, CNRS and ENS, Paris; Mitchell Canham, University of Alberta

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[2]FGB & VP, *Comptes Rendus Mécanique* **342**, 79-84 (2014)
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Outline

1. Introduction to variational discretization methods
2. Problem formulation: tubes conveying fluid
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4. Variational derivation of tube-fluid equations
5. Discretization in space, continuous time
6. Dynamic behavior of simple models
7. Discretization in space and time
8. Conclusions and open questions

A brief intro to variational method

- Standard numerical methods for Euler-Lagrange equations (e.g. Runge-Kutta), as a rule, do not preserve *any* of the integrals of motion for mechanical systems. These integrals are especially important for studying long-term stability of mechanical systems.
- Instead, consider a mechanical system with configuration space $\mathbb{R}^n$ and discretization of the trajectory $t_0, t_1, \dots, t_n$ and $\boldsymbol{q}(t_i) \simeq \boldsymbol{q}_i \in \mathbb{R}^n$. Write the discrete Lagrangian
$$L(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t) \to L(\boldsymbol{q}_i, \dot{\boldsymbol{q}}_i, t_i) \simeq L\left(\boldsymbol{q}_i, \frac{\boldsymbol{q}_{i+1} - \boldsymbol{q}_i}{h}\right) = L_i(\boldsymbol{q}_i, \boldsymbol{q}_{i+1})$$
- The action becomes (D_i is the derivative wrt the i-th argument).
$$S = \sum_i L_i(\boldsymbol{q}_i, \boldsymbol{q}_{i+1}) \quad \Rightarrow \quad \delta S = \sum_i (D_1 L_i + D_2 L_{i-1}) \, \delta \boldsymbol{q}_i = 0$$
- Discrete Euler-Lagrange equations $\quad D_1 L_i + D_2 L_{i-1} = \mathbf{0}$
- In general, appropriately defined momenta are conserved by variational methods with machine precision.
- Energy is not conserved, but it is oscillatory and is typically preserved on average with a very high accuracy.

Advantages and disadvantages of variational integrators

- Variational methods preserve symplectic structure, momenta, Noether theorems, long time energy stability, can incorporate constraints ...

+ Conservation laws preserved to machine precision for any time step
+ Preservation of symplectic structure of the system
+ No artificial momentum and energy sources and sinks, advantages for stability of and long-term behavior study

– Except for some simple cases, the integrators are implicit and (slightly) more complex to use than *e.g.* Runge-Kutta methods.

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– Except for some simple cases, the integrators are implicit and (slightly) more complex to use than *e.g.* Runge-Kutta methods.

+ Potential for application to fluid-structure interactions,

+/- To use these methods, fluid-structure interaction needs to be expressed in the variational form

Example: variational treatment of a tube conveying fluid

Figure: Image of a garden hose and its mathematical description

- No friction in the system, incompressible fluid, Reynolds numbers $\sim 10^4$ (much higher in some applications), general 3D motions
- Hose can stretch and bend arbitrarily (inextensible also possible)
- Cross-section of the hose changes dynamically with deformations: *collapsible tube*

Previous work

- *Constant fluid velocity in the tube*, 2D dynamics:
  English: Benjamin (1961); Gregory, Païdoussis (1966); Païdoussis (1998); Doare, De Langre (2002); Flores, Cros (2009), ...
  Russian: Bolotin (?) (1956), Svetlitskii (monographs 1982, 1987), Danilin (2005), Zhermolenko (2008), Akulenko *et al.* (2015) ...
  Hard to generalize to general 3D motions
  Not possible to consistently incorporate the cross-sectional dynamics
- Elastic rod with directional (tangent) momentum source at the end – the follower-force method, see Bou-Rabee, Romero, Salinger (2002), critiqued by Elishakoff (2005).
- Shell models: Paidoussis & Denise (1972), Matsuzaki & Fung (1977), Heil (1996), Heil & Pedley (1996) , ...: Complex, computationally intensive, difficult (impossible) to perform analytic work for non-straight tubes.
- 3D dynamics from Cosserat's model (Beauregard, Goriely & Tabor 2010): Force balance, not variational, cannot accommodate dynamical change of the cross-section.
- Variational derivation: FGB & VP (2014,2015).

Variational treatment of changing cross-sections dynamics

Mathematical preliminaries:

- Rod dynamics is described by $SE(3)$-valued functions (rotations and translations in space) $\pi(s,t) = (\Lambda, \mathbf{r})(s,t)$.
- Fluid dynamics inside the rod is described by 1D diffeomorphisms $s = \varphi(a,t)$, where a is the Lagrangian label.
- Conservation of 1-form volume element (fluid incompressibility) defined through a holonomic constraint:

$$Q(\boldsymbol{\Omega}, \boldsymbol{\Gamma}) := A \left| \frac{d\mathbf{r}}{ds} \right| = \left(Q_0 \circ \varphi^{-1}(s,t) \right) \partial_s \varphi^{-1}(s,t) \qquad (1)$$

- Alternatively, evolution equation for Q is $\partial_t Q + \partial_s (Qu) = 0$.
- Note that commonly used $Au =$const does not conserve volume for time-dependent flow. See *e.g.* [Kudryashov *et al*, Nonlinear dynamics (2008)] for correct derivation in 1D.

Mathematical preliminaries: Geometric rod theory for elastic rods I

- Purely elastic Lagrangian

$$\mathcal{L} = \mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}, \mathbf{r}', \Lambda, \dot{\Lambda}, \Lambda')$$

- Use $SE(3)$ symmetry reduction [Simo, Marsden, Krishnaprasad 1988] (SMK) to reduce the Lagrangian to $\ell(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma})$ of the following coordinate-invariant variables (prime$= \partial_s$, dot$=\partial_t$):

$$\boldsymbol{\Gamma} = \Lambda^{-1}\mathbf{r}' \,, \; \boldsymbol{\Omega} = \Lambda^{-1}\Lambda' \,, \tag{2}$$

$$\boldsymbol{\gamma} = \Lambda^{-1}\dot{\mathbf{r}} \,, \; \boldsymbol{\omega} = \Lambda^{-1}\dot{\Lambda} \,. \tag{3}$$

- Note that symmetry reduction for elastic rods is *left-invariant* (reduces to body variables).
- Notation: small letters (e.g. $\boldsymbol{\omega}, \boldsymbol{\gamma}$) denote time derivatives; capital letters (e.g. $\boldsymbol{\Omega}, \boldsymbol{\Gamma}$) denote the s-derivatives.

Mathematical preliminaries: Geometric rod theory for elastic rods II

- Euler Poincaré theory: [Holm, Marsden, Ratiu 1998].
For elastic rods: compute variations as in [Ellis, Holm, Gay-Balmaz, VP and Ratiu, *Arch. Rat.Mech. Anal.*, (2010)]: consider $\boldsymbol{\Sigma} = \Lambda^{-1}\delta\Lambda \in \mathfrak{so}(3)$ and $\boldsymbol{\Psi} = \Lambda^{-1}\delta \mathbf{r} \in \mathbb{R}^3$, and $(\boldsymbol{\Sigma}, \boldsymbol{\Psi}) \in \mathfrak{se}(3)$.

$$\delta\boldsymbol{\omega} = \frac{\partial \boldsymbol{\Sigma}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{\Sigma}, \qquad \delta\boldsymbol{\gamma} = \frac{\partial \psi}{\partial t} + \boldsymbol{\gamma} \times \boldsymbol{\Sigma} + \boldsymbol{\omega} \times \psi \tag{4}$$

$$\delta\boldsymbol{\Omega} = \frac{\partial \boldsymbol{\Sigma}}{\partial s} + \boldsymbol{\Omega} \times \boldsymbol{\Sigma}, \qquad \delta\boldsymbol{\Gamma} = \frac{\partial \psi}{\partial s} + \boldsymbol{\Gamma} \times \boldsymbol{\Sigma} + \boldsymbol{\Omega} \times \psi, \tag{5}$$

- Compatibility conditions (cross-derivatives in s and t are equal)

$$\boldsymbol{\Omega}_t - \boldsymbol{\omega}_s = \boldsymbol{\Omega} \times \boldsymbol{\omega}\,, \quad \boldsymbol{\Gamma}_t + \boldsymbol{\omega} \times \boldsymbol{\Gamma} = \boldsymbol{\gamma}_s + \boldsymbol{\Omega} \times \boldsymbol{\gamma}\,.$$

- Critical action principle $\delta \int \ell \mathrm{d}t\mathrm{d}s = 0+$ (4,5) give SMK equations.

$$0 = \delta \int \ell \mathrm{d}t\mathrm{d}s = \int \left\langle \frac{\delta \ell}{\delta \boldsymbol{\omega}}\,, \delta\boldsymbol{\omega} \right\rangle + \int \left\langle \frac{\delta \ell}{\delta \boldsymbol{\Omega}}\,, \delta\boldsymbol{\Omega} \right\rangle + \dots$$

$$= \int \langle \text{linear momentum eq}, \boldsymbol{\Psi} \rangle + \langle \text{angular momentum eq}, \boldsymbol{\Sigma} \rangle \, \mathrm{d}t\mathrm{d}s$$

Mathematics preliminaries: right-invariant incompressible fluid motion

- Following Arnold (1966), describe a 3D incompressible fluid motion by Diff_{Vol} group $\mathbf{r} = \varphi(\mathbf{a}, t)$.
- Eulerian fluid velocity is $u = \varphi_t \circ \varphi^{-1}$; symmetry-reduced Lagrangian is $\ell = 1/2 \int u^2 \mathrm{d}\mathbf{r}$.
- Variations of velocity are computed as

$$\eta = \delta\varphi \circ \varphi^{-1}(s,t)\,, \quad \delta u = \eta_t + u\nabla\eta - \eta\nabla u\,. \qquad (6)$$

- Incompressibility condition

$$J = \left|\frac{\partial \mathbf{r}}{\partial \mathbf{a}}\right| = 1 \Rightarrow \text{Lagrange multiplier } p\,. \qquad (7)$$

- Euler equations: $\delta \int \ell \, \mathrm{d}V \, \mathrm{d}t = 0$ with (6) and (7)

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p\,, \quad \mathrm{div}\, u = 0$$

Garden hoses: Lagrangian and symmetry reductions

1. Symmetry group of the system (ignoring gravity for now)
$$G = SE(3) \times \mathrm{Diff}_A(\mathbb{R}) = SO(3) \circledS \mathbb{R} \times \mathrm{Diff}_A(\mathbb{R}) \,. \qquad (8)$$
2. Position of elastic tube and fluid:
$$(\pi, \varphi) \cdot \Big((\Lambda_0, \mathbf{r}_{t,0}), \mathbf{r}_f \Big) = \Big(\underbrace{\pi \cdot (\Lambda_0, \mathbf{r}_{t,0})}_{\text{left invariant}}, \underbrace{\pi \cdot \mathbf{r}_f \circ \varphi^{-1}(s,t)}_{\text{right invariant}} \Big) \,.$$
3. Velocities:
$$\begin{aligned}(\mathbf{v}_r \,,\, \mathbf{v}_f) =& \frac{d}{dt}\Big(\mathbf{r}(s,t) \,,\, \mathbf{r} \circ \varphi^{-1}(s,t) \Big) \\ =& \Big(\dot{\mathbf{r}}(s,t), \dot{\mathbf{r}} \circ \varphi^{-1}(s,t) + \mathbf{r}'(s,t) u(s,t) \Big) \,. \end{aligned} \qquad (9)$$
4. Change in cross-section $A = A(\boldsymbol{\Omega}, \boldsymbol{\Gamma})$
5. Incompressibility condition $J = A(s,t) \frac{\partial a}{\partial s} |\boldsymbol{\Gamma}| = 1$ with Lagrange multiplier μ (pressure)
$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial s}(Qu) = 0 \,, \quad \text{with} \quad Q = A|\boldsymbol{\Gamma}| \,. \qquad (10)$$

Equations of motion

$$
\begin{cases}
(\partial_t + \boldsymbol{\omega}\times)\dfrac{\delta \ell}{\delta \boldsymbol{\omega}} + \boldsymbol{\gamma} \times \dfrac{\delta \ell}{\delta \boldsymbol{\gamma}} + (\partial_s + \boldsymbol{\Omega}\times)\left(\dfrac{\delta \ell}{\delta \boldsymbol{\Omega}} - \dfrac{\partial Q}{\partial \boldsymbol{\Omega}}\mu\right) + \boldsymbol{\Gamma} \times \left(\dfrac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \dfrac{\partial Q}{\partial \boldsymbol{\Gamma}}\mu\right) = 0 \\
(\partial_t + \boldsymbol{\omega}\times)\dfrac{\delta \ell}{\delta \boldsymbol{\gamma}} + (\partial_s + \boldsymbol{\Omega}\times)\left(\dfrac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \dfrac{\partial Q}{\partial \boldsymbol{\Gamma}}\mu\right) = 0 \\
m_t + \partial_s (mu - \mu) = 0, \quad m := \dfrac{1}{Q}\dfrac{\delta \ell}{\delta u} \\
\partial_t Q + \partial_s (Qu) = 0, \quad Q = A|\boldsymbol{\Gamma}| \\
\partial_t \boldsymbol{\Omega} = \boldsymbol{\omega} \times \boldsymbol{\Omega} + \partial_s \boldsymbol{\omega}\,, \qquad \partial_t \boldsymbol{\Gamma} + \boldsymbol{\omega} \times \boldsymbol{\Gamma} = \partial_s \boldsymbol{\gamma} + \boldsymbol{\Omega} \times \boldsymbol{\gamma}
\end{cases}
$$

Assume $A = A(\boldsymbol{\Omega}, \boldsymbol{\Gamma})$, symmetric tube with axis $\mathbf{E}_1$ for Lagrangian

$$
\ell(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, u) \\
= \frac{1}{2}\int \left(\alpha|\boldsymbol{\gamma}|^2 + \left\langle \mathbb{I}\boldsymbol{\omega}, \boldsymbol{\omega}\right\rangle + \rho A(\boldsymbol{\Omega}, \boldsymbol{\Gamma})|\boldsymbol{\gamma} + \boldsymbol{\Gamma} u|^2 - \left\langle \mathbb{J}\boldsymbol{\Omega}, \boldsymbol{\Omega}\right\rangle - \lambda|\boldsymbol{\Gamma} - \mathbf{E}_1|^2\right)|\boldsymbol{\Gamma}|\mathrm{d}s\,.
$$

See FGB & VP for linear stability analysis, nonlinear solutions *etc.*

Non-conservation of energy

Define the energy function

$$e(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, u) = \int_0^L \left(\frac{\delta \ell}{\delta \boldsymbol{\omega}} \cdot \boldsymbol{\omega} + \frac{\delta \ell}{\delta \boldsymbol{\gamma}} \cdot \boldsymbol{\gamma} + \frac{\delta \ell}{\delta u} u \right) \mathrm{d}s - \ell(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, u)$$

and boundary forces at the exit (free boundary)

$$F_u := \frac{\delta \ell}{\delta u} u - \mu Q \Big|_{s=L}, \quad \mathbf{F}_{\boldsymbol{\Gamma}} := \frac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \mu \frac{\partial Q}{\partial \boldsymbol{\Gamma}} \Big|_{s=L}, \quad \mathbf{F}_{\boldsymbol{\Omega}} := \frac{\delta \ell}{\delta \boldsymbol{\Omega}} - \mu \frac{\partial Q}{\partial \boldsymbol{\Omega}} \Big|_{s=L}.$$

Then, the energy changes according to

$$\frac{d}{dt} e(\boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, u) = \int_0^T \left(\mathbf{F}_{\boldsymbol{\Omega}} \cdot \boldsymbol{\Omega} + \mathbf{F}_{\boldsymbol{\Gamma}} \cdot \boldsymbol{\Gamma} + F_u u \right) \Big|_{s=L}^{s=0} \mathrm{d}t.$$

The system is not closed and the energy is not conserved. Similar statement is true for the discrete version of the problem.

Variational discretization of tube conveying fluid in space: definitions

- As in Demoures *et al* (2014), discretize s as $s \to (s_0, s_1, \ldots, s_N)$ and define the variables $\lambda_i := \Lambda_i^{-1}\Lambda_{i+1} \in SO(3)$ (relative orientation) and $\boldsymbol{\kappa}_i = \Lambda_i^{-1}(\mathbf{r}_{i+1} - \mathbf{r}_i) \in \mathbb{R}^3$ (relative shift).
- Define the forward Lagrangian map $s = \varphi(a, t)$ and back to labels map $a = \psi(s, t) = \varphi^{-1}(s, t)$.
- Discretize $\psi(s, t)$ as $\overline{\psi}(t) = (\psi_1(t), \psi_2(t), \ldots, \psi_N(t))$ with $\psi_i(t) \simeq \psi(s_i, t)$.
- Discretize the spatial derivative as $D_i\overline{\psi}(t) := \sum_{j \in J} a_j \psi_{i+j}(t)$, where J is a discrete set around 0,
- For example, we can take $D_i\overline{\psi} = (\psi_i - \psi_{i-1})/h$ (backwards derivative), in that case

$$J = (-1, 0) \quad \text{and} \quad a_{-1} = -\frac{1}{h}\, , a_0 = \frac{1}{h}\, .$$

- For more general cases, for example, variable s-step, we take $D_i\overline{\psi}(t) := \sum_{j \in i+J} A_{ij}\psi_j(t)$.

Variational discretization of a tube conveying fluid in space: definitions

- Discretize the conservation law $(Q_0 \circ \varphi^{-1})\partial_s \varphi^{-1} = Q(\boldsymbol{\Omega}, \boldsymbol{\Gamma})$ as

$$Q_0 D_i \overline{\psi} = F(\lambda_i, \boldsymbol{\kappa}_i) := F_i \quad \Rightarrow \quad \dot{F}_i + D_i \left(\overline{uF} \right) = 0$$

- Differentiate the identity $s = \varphi(\psi(s,t), t)$ with respect to time to get $u(s,t) = (\varphi_t \circ \psi)(s,t)$ as

$$u(s,t) = (\partial_t \varphi \circ \psi)(s,t) = -\frac{\partial_t \psi(s,t)}{\partial_s \psi(s,t)} \quad \Rightarrow \quad u_i(t) = -\frac{\dot{\psi}_i}{D_i \overline{\psi}}$$

- Define the approximation for the action

$$S = \int \ell(\boldsymbol{\omega}, \gamma, \boldsymbol{\Omega}, \boldsymbol{\Gamma}, u) \mathrm{d}t \mathrm{d}s \to S_d = \int \sum_i \ell_d(\boldsymbol{\omega}_i, \gamma_i, \lambda_i, \boldsymbol{\kappa}_i, u_i) \mathrm{d}t$$

Variational discretization of variables: variations

- Define the discrete action principle

$$\delta \int \sum_i \left[\ell_d(\boldsymbol{\omega}_i, \boldsymbol{\gamma}_i, \lambda_i, \boldsymbol{\kappa}_i, u_i) + \mu_i \left(Q_0 D_i \overline{\psi} - F(\lambda_i, \boldsymbol{\kappa}_i) \right) \right] \mathrm{d}t = 0$$

- Compute the variations of elastic in variables terms of free variations $\xi_i = \Lambda_i^{-1} \delta \Lambda_i \in \mathfrak{so}(3)$ and $\boldsymbol{\eta}_i = \Lambda_i^{-1} \delta \mathbf{r}_i \in \mathbb{R}^3$ as

$$\delta \lambda_i = -\xi_i \lambda_i + \lambda_i \xi_{i+1} \quad \delta \boldsymbol{\kappa}_i = -\boldsymbol{\xi}_i \times \boldsymbol{\kappa}_i + \lambda_i \boldsymbol{\eta}_{i+1} - \boldsymbol{\eta}_i \,,$$

- Compute the variations of velocity in terms of $\delta \psi_i$

$$\delta u_i = -\frac{\delta \dot{\psi}_i}{D_i \overline{\psi}} + \frac{\dot{\psi}_i}{(D_i \overline{\psi})^2} \sum_{j \in J} a_j \delta \psi_{i+j} = -\frac{Q_0}{D_i \overline{\psi}} \left(\delta \dot{\psi}_i + u_i D_i \overline{\delta \psi} \right) \,.$$

- Terms proportional to ξ_i give angular momentum conservation law
- Terms proportional to $\boldsymbol{\eta}_i$ give linear momentum conservation law
- Terms proportional to ψ_i give a fluid momentum, but we need to use the fluid conservation law $Q_0 D_i \overline{\psi} = F(\lambda_i, \boldsymbol{\kappa}_i) := F_i$ to remove all $\overline{\psi}$ from equations.

Variational integrator for spatial discretization I

- Angular momentum: terms proportional to $\boldsymbol{\xi}_i = \left(\Lambda_i^{-1}\delta\Lambda_i\right)^\vee$ [1]

$$\left(\frac{d}{dt} + \boldsymbol{\omega}_i\times\right)\frac{\partial \ell_d}{\partial \boldsymbol{\omega}_i} + \boldsymbol{\gamma}_i \times \frac{\partial \ell_d}{\partial \boldsymbol{\gamma}_i} + \left[\left(\frac{\partial \ell_d}{\partial \lambda_i} - \mu_i \frac{\partial F}{\partial \lambda_i}\right)\lambda_i^T - \lambda_{i-1}^T\left(\frac{\partial \ell_d}{\partial \lambda_{i-1}} - \mu_{i-1}\frac{\partial F}{\partial \lambda_{i-1}}\right)\right]^\vee + \boldsymbol{\kappa}_i \times \left(\frac{\partial \ell_d}{\partial \boldsymbol{\kappa}_i} - \mu_i \frac{\partial F}{\partial \boldsymbol{\kappa}_i}\right) = \mathbf{0}$$

Compare with the continuum equation:

$$(\partial_t + \boldsymbol{\omega}\times)\frac{\delta \ell}{\delta \boldsymbol{\omega}} + \boldsymbol{\gamma}\times\frac{\delta \ell}{\delta \boldsymbol{\gamma}} + (\partial_s + \boldsymbol{\Omega}\times)\left(\frac{\delta \ell}{\delta \boldsymbol{\Omega}} - \frac{\partial Q}{\partial \boldsymbol{\Omega}}\mu\right) + \boldsymbol{\Gamma}\times\left(\frac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \frac{\partial Q}{\partial \boldsymbol{\Gamma}}\mu\right) = \mathbf{0}$$

[1] We denote $\widehat{\mathbf{a}} = -\epsilon_{ijk}\mathbf{a}_k$ is the hat map for $\mathbb{R}^3 \to \mathfrak{so}(3)$, and $a^\vee = \mathbf{a} \in \mathbb{R}^3$ is its inverse

Variational integrator for spatial discretization I

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$$\left(\frac{d}{dt}+\boldsymbol{\omega}_i\times\right)\frac{\partial \ell_d}{\partial \boldsymbol{\omega}_i}+\boldsymbol{\gamma}_i\times\frac{\partial \ell_d}{\partial \boldsymbol{\gamma}_i}+\left[\left(\frac{\partial \ell_d}{\partial \lambda_i}-\mu_i\frac{\partial F}{\partial \lambda_i}\right)\lambda_i^T\right.$$
$$\left.-\lambda_{i-1}^T\left(\frac{\partial \ell_d}{\partial \lambda_{i-1}}-\mu_{i-1}\frac{\partial F}{\partial \lambda_{i-1}}\right)\right]^\vee+\boldsymbol{\kappa}_i\times\left(\frac{\partial \ell_d}{\partial \boldsymbol{\kappa}_i}-\mu_i\frac{\partial F}{\partial \boldsymbol{\kappa}_i}\right)=\mathbf{0}$$

Compare with the continuum equation:

$$(\partial_t+\boldsymbol{\omega}\times)\frac{\delta \ell}{\delta \boldsymbol{\omega}}+\boldsymbol{\gamma}\times\frac{\delta \ell}{\delta \boldsymbol{\gamma}}+(\partial_s+\boldsymbol{\Omega}\times)\left(\frac{\delta \ell}{\delta \boldsymbol{\Omega}}-\frac{\partial Q}{\partial \boldsymbol{\Omega}}\mu\right)+\boldsymbol{\Gamma}\times\left(\frac{\delta \ell}{\delta \boldsymbol{\Gamma}}-\frac{\partial Q}{\partial \boldsymbol{\Gamma}}\mu\right)=\mathbf{0}$$

- Linear momentum: terms proportional to $\boldsymbol{\eta}_i = \Lambda_i^{-1}\delta\mathbf{r}_i$

$$\left(\frac{d}{dt}+\boldsymbol{\omega}_i\times\right)\frac{\partial \ell_d}{\partial \boldsymbol{\gamma}_i}+\left(\frac{\partial \ell_d}{\partial \boldsymbol{\kappa}_i}-\mu_i\frac{\partial F}{\partial \boldsymbol{\kappa}_i}\right)-\lambda_{i-1}^T\left(\frac{\partial \ell_d}{\partial \boldsymbol{\kappa}_{i-1}}-\mu_{i-1}\frac{\partial F}{\partial \boldsymbol{\kappa}_{i-1}}\right)=\mathbf{0}$$

Corresponding continuum equation

$$(\partial_t+\boldsymbol{\omega}\times)\frac{\delta \ell}{\delta \boldsymbol{\gamma}}+(\partial_s+\boldsymbol{\Omega}\times)\left(\frac{\delta \ell}{\delta \boldsymbol{\Gamma}}-\frac{\partial Q}{\partial \boldsymbol{\Gamma}}\mu\right)=\mathbf{0}$$

---

[1] We denote $\widehat{\mathbf{a}} = -\epsilon_{ijk}\mathbf{a}_k$ is the hat map for $\mathbb{R}^3 \to \mathfrak{so}(3)$, and $a^\vee = \mathbf{a} \in \mathbb{R}^3$ is its inverse

Variational integrator for spatial discretization II

- Fluid momentum equation: terms proportional to $\delta\psi_i$

$$\frac{d}{dt}\left(\frac{1}{F_i}\frac{\partial \ell_d}{\partial u_i}\right) + D_i^+\left(\overline{\frac{u}{F}\frac{\partial \ell_d}{\partial u}} - \overline{\mu}\right) = 0$$

where we have defined the dual discrete derivative $D_i^+\overline{X} := -\sum_{j\in J} a_j X_{i-j}$, and $m^\vee{}_c := -\frac{1}{2}\sum_{ab}\epsilon_{abc}m_{ab}$

Continuum equation:

$$m_t + \partial_s\left(mu - \mu\right) = 0, \quad m := \frac{1}{Q}\frac{\delta \ell}{\delta u}$$

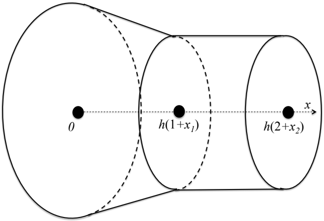
- Conservation law in the discrete form:

$$Q_0 D_i\overline{\psi} = F(\lambda_i, \boldsymbol{\kappa}_i) := F_i \quad \Rightarrow \quad \dot{F}_i + D_i\left(\,\overline{uF}\,\right) = 0$$

Continuum version

$$Q(\boldsymbol{\Omega}, \boldsymbol{\Gamma}) := A\,|\boldsymbol{\Gamma}| = \left(Q_0 \circ \varphi^{-1}(s,t)\right)\,\varphi' \circ \varphi^{-1}(s,t) \Rightarrow \partial_t Q + \partial_s(Qu) = 0$$

An example: 1D stretching motion



- Assume that all motion of the tube is along the $\mathbf{E}_1$ direction, so $\mathbf{r}_k = h(k + x_k, 0, 0)^T$ and $\Lambda_i = \mathrm{Id}_{3\times 3}$, where x_k is the dimensionless deviation from equilibrium.
- Consider a simplified model with only three points, $k = 0, 1, 2$, denote $x = x_1$.
- Fixed BC on the left, $x_0 = 0$ and no deformation in the cross-section.
- Free BC on the right, $x_2 = x_1 = x$.
- Express all variables u_i, μ_i in terms of x_i and its time derivatives.
- Get a nonlinear ODE $\ddot{x} = f(x, \dot{x})$ for a single variable $x(t)$.

Numerical solutions of stretching tube equations

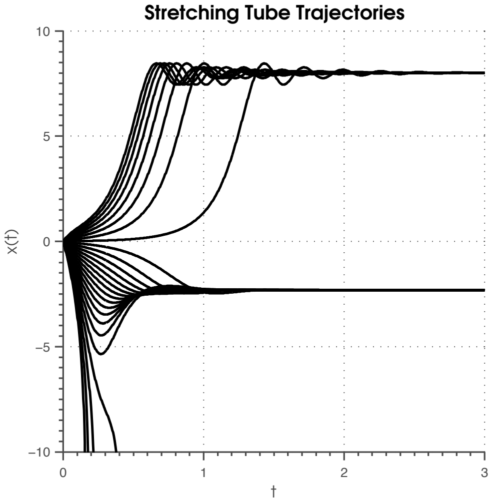


Figure: Trajectories $x(t)$ starting with $x(0) = 0$ for varying initial conditions $x'(0) = x'_0$.

Steady states and their stability as a function of u_0

Parameter values:

$$h = 0.1,\ T = 1,\ \mu_0 = 1,\ \rho = 11,\ F_1 = 2,\ \alpha = 1,\ \beta = 3,\ \xi = 1.$$

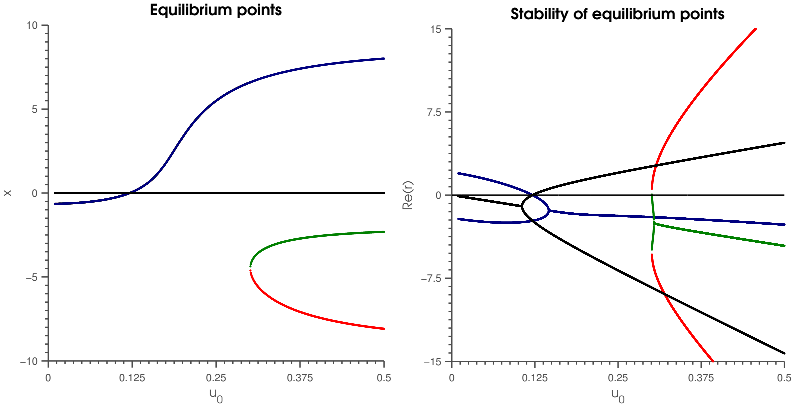


Figure: Left: Equilibrium points as a function of u_0, Right: their stability. Color labeling is the same for each equilibrium point.

Time and space discretization

- Discretize $s \to (s_0, s_1, \ldots, s_N)$ and $t \to (t_0, t_1, \ldots, t_M)$.
- Define the temporal and spatial relative orientations and shifts (first index is s, second index is t):

$$\lambda_{i,j} := \Lambda_{i,j}^{-1}\Lambda_{i+1,j}\,, \quad \boldsymbol{\kappa}_{i,j} := \Lambda_{i,j}^{-1}\left(\mathbf{r}_{i+1,j} - \mathbf{r}_{i,j}\right)$$
$$q_{i,j} := \Lambda_{i,j}^{-1}\Lambda_{i,j+1}\,, \quad \gamma_{i,j} := \Lambda_{i,j}^{-1}\left(\mathbf{r}_{i,j+1} - \mathbf{r}_{i,j}\right)\,.$$

- Define discrete spatial and temporal derivatives are $D^s_{i,j}\overline{\psi} := \sum_{k\in K} a_j \psi_{i,j+k}\,, \quad D^t_{i,j}\overline{\psi} := \sum_{m\in M} b_m \psi_{i+m,j}$
- The velocity is given by

$$u_{i,j} = -\frac{D^t_{i,j}\overline{\psi}}{D^s_{i,j}\overline{\psi}} \qquad \left(\text{Compare with} \quad u = -\frac{\psi_t}{\psi_s}\right)$$

- Discrete conservation law is

$$Q_0 D^s_{i,j}\overline{\psi} = F_{i,j} \quad \Rightarrow \quad D^t_{i,j}\overline{F} + D^s_{i,j}(\,\overline{uF}\,) = 0\,.$$

Variational integrator in time and space

- Consider the critical discrete action principle
$$\delta \sum_{i,j} \mathcal{L}_d \left(\lambda_{i,j}, \boldsymbol{\kappa}_{i,j}, q_{i,j}, \gamma_{i,j}, u_{i,j}\right) + \mu_{i,j} \left(Q_0 D^s_{i,j} \overline{\psi} - F(\lambda_{i,j}, \boldsymbol{\kappa}_{i,j})\right) = 0$$
- Perform variations to obtain equations of motion
- Angular momentum equation: terms proportional to
$\boldsymbol{\Sigma}_{i,j} = \left(\Lambda^{-1}_{i,j}\delta\Lambda_{i,j}\right)^\vee$

$$\left[\frac{\partial \mathcal{L}_d}{\partial q_{i,j}} q^T_{i,j} - q^T_{i,j-1}\frac{\partial \mathcal{L}_d}{\partial q_{i,j-1}}\right]^\vee + \left[\left(\frac{\partial \mathcal{L}_d}{\partial \lambda_{i,j}} - \mu_{i,j}\frac{\partial F}{\partial \lambda_{i,j}}\right)\lambda^T_{i,j} - \lambda^T_{i-1,j}\left(\frac{\partial \mathcal{L}_d}{\partial \lambda_{i-1,j}} - \mu_{i-1,j}\frac{\partial F}{\partial \lambda_{i-1,j}}\right)\right]^\vee + \boldsymbol{\gamma}_{i,j} \times \frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\gamma}_{i,j}} + \boldsymbol{\kappa}_{i,j} \times \frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\kappa}_{i,j}} = \mathbf{0}$$

Continuum equation for reference

$$(\partial_t + \boldsymbol{\omega}\times)\frac{\delta \ell}{\delta \boldsymbol{\omega}} + \boldsymbol{\gamma} \times \frac{\delta \ell}{\delta \boldsymbol{\gamma}} + (\partial_s + \boldsymbol{\Omega}\times)\left(\frac{\delta \ell}{\delta \boldsymbol{\Omega}} - \frac{\partial Q}{\partial \boldsymbol{\Omega}}\mu\right) + \boldsymbol{\Gamma} \times \left(\frac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \frac{\partial Q}{\partial \boldsymbol{\Gamma}}\mu\right) = \mathbf{0}$$

Equations of motion, continued

- Linear momentum equation: terms proportional to $\boldsymbol{\Psi}_{i,j} = \Lambda_{i,j}^{-1} \delta \mathbf{r}_{i,j}$

$$\frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\gamma}_{i,j}} - q_{i,j-1}^T \frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\gamma}_{i,j-1}} + \left(\frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\kappa}_{i,j}} - \mu_{i,j} \frac{\partial F}{\partial \boldsymbol{\kappa}_{i,j}} \right) - \lambda_{i-1,j}^T \left(\frac{\partial \mathcal{L}_d}{\partial \boldsymbol{\kappa}_{i-1,j}} - \mu_{i-1,j} \frac{\partial F}{\partial \boldsymbol{\kappa}_{i-1,j}} \right) = \mathbf{0}$$

Continuum version for reference:

$$(\partial_t + \boldsymbol{\omega} \times) \frac{\delta \ell}{\delta \boldsymbol{\gamma}} + (\partial_s + \boldsymbol{\Omega} \times) \left(\frac{\delta \ell}{\delta \boldsymbol{\Gamma}} - \frac{\partial Q}{\partial \boldsymbol{\Gamma}} \mu \right) = \mathbf{0}$$

- Fluid momentum equation: terms proportional to $\delta \psi_{i,j}$

$$D_{i,j}^{t,+} \overline{m} + D_{i,j}^{s,+} \left(\overline{um} - \overline{\mu} \right) = 0 \,, \quad m_{i,j} := \frac{1}{F_{i,j}} \frac{\partial \mathcal{L}_d}{\partial u_{i,j}}$$

$$D_{i,j}^{s,+} \overline{X} := - \sum_{k \in K} a_k X_{i,j-k} \,, \quad D_{i,j}^{t,+} \overline{X} := - \sum_{m \in M} b_j X_{i-m,j}$$

Continuum version: $\quad m_t + \partial_s (mu - \mu) = 0, \quad m := \dfrac{1}{Q} \dfrac{\delta \ell}{\delta u}$

Future work

1. Change in tube's radius $R(s, t)$ dynamically coupled with tube+fluid motion in 3D (FGB & VP, in preparation)
2. Understanding the discretization of free end boundary conditions
3. Variational spectral methods – *expansion in modes* - if possible.
4. Stability of variational FSI methods
5. Variational discretization of non-closed systems
6. Simulation of bending motion in 2D and comparison with linearized theory
7. 1D reduction and comparison with exact solutions
8. Suggestions welcome for other examples of fluid-structure interactions treatable by this method

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9. Workshop on computer graphics applications (funding for workshop awarded from PIMS, tentative timing – spring of 2018): What conservation laws are needed for graphics to 'look good'