On convergence of discrete exterior calculus

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Connections in geometric numerical integration and structure-preserving discretization
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Plan of the talk

Background:

- Discrete exterior calculus
- Previous work on the convergence problem

Our results to date on

- Consistency
- Convergence in H^1, and in L^2

What is DEC?

Main idea: DEC is a framework for constructing discrete versions of exterior differential objects (Desbrun, Hirani, Leok, and Marsden 2005/2003; Hirani 2003).

- General relativity (Frauendiener 2006)
- Electrodynamics (Stern, Tong, Desbrun, Marsden 2007)
- Linear elasticity (Yavari 2008)
- Computational modeling (Desbrun, Kanso, Tong 2008)
- Port-Hamiltonian systems (Seslija, Schaft, Scherpen 2012)
- Digital geometry processing (Crane, de Goes, Desbrun, Schröder 2013)
- Darcy flow (Hirani, Nakshatrala, Chaudhry 2015)
- Navier-Stokes equations (Mohamed, Hirani, Samtaney 2016)

Codifferential

We define the coderivative $\delta : \Omega^k \to \Omega^{k-1}$ as the L^2-adjoint of d.

$$\langle \delta\alpha, \beta \rangle = \langle \alpha, \mathrm{d}\beta \rangle$$

Note $\delta\delta = 0$ and $\{0\} \rightleftarrows \Omega^0 \underset{\delta}{\overset{\mathrm{d}}{\rightleftarrows}} \Omega^1 \rightleftarrows \ldots \rightleftarrows \Omega^n \rightleftarrows \{0\}$.

For k-forms, we have

$$\delta = (-1)^{n(k-1)+1} \star \mathrm{d} \star = (-1)^k \star^{-1} \mathrm{d} \star$$

$\delta = -\mathrm{div}$	for 1-forms in 3D
$\delta = \mathrm{curl}$	for 2-forms in 3D
$\delta = -\mathrm{grad}$	for 3-forms in 3D
$\delta = -\mathrm{div}$	for 1-forms in 2D
$\delta = \mathrm{grad}^\perp = -J \circ \mathrm{grad}$	for 2-forms in 2D

The Hodge-Laplace operator

The Hodge-Laplace operator is $\Delta = \delta \mathrm{d} + \mathrm{d}\delta$.

$\Delta = -\operatorname{div}\operatorname{grad}$	for 0-forms
$\Delta = \operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div}$	for 1-forms in 3D
$\Delta = -\operatorname{grad}\operatorname{div} + \operatorname{curl}\operatorname{curl}$	for 2-forms in 3D
$\Delta = -\operatorname{div}\operatorname{grad}$	for 3-forms in 3D
$\Delta = \operatorname{grad}^{\perp}\operatorname{rot} - \operatorname{grad}\operatorname{div}$	for 1-forms in 2D
$\Delta = -\operatorname{div}\operatorname{grad}$	for 2-forms in 2D

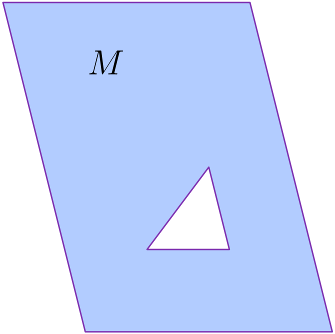
The Hodge-Laplacian Poisson problem

Consider the problem

$$\Delta u \equiv (\delta \mathrm{d} + \mathrm{d}\delta) u = f$$

to find $u \in \Omega^k(M)$, where

- $M \subset \mathbb{R}^n$ is bdd, polyhedral domain
- $f \in \Omega^k(M)$ is given
- Some boundary condition is needed

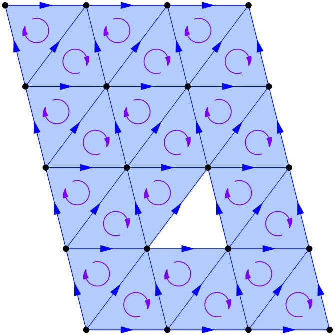


Discrete domain

- A *k-simplex* in $\mathbb{R}^n$ is the k-dimensional convex span $\sigma = [v_0, ..., v_k]$ of $(k+1)$ affinely independent vertices. A simplicial n-complex K is a collection of n-simplices such that:
  - i Every face of a simplex in K is in K;
  - ii The intersection of any two simplices of K is either empty or a face of both.

- A triangulation of a domain in $\mathbb{R}^n$ is a simplicial complex K_h of the same dimension satisfying

$$\bigcup_{\sigma \in \Delta_n(K_h)} \sigma = M$$



Chains and cochains

- A k-chain $\in C_k(K_h)$ is a finite formal sum

$$\gamma = A_1\sigma_1 + A_2\sigma_2 + \ldots A_m\sigma_m$$

  of k-simplices, where A_i are real coefficients.
- A discrete k-form is a k-cochain $\in C^k(K_h) = \mathrm{Hom}(C_k(K_h), \mathbb{R})$.

Given a basis $\{\sigma_i\}$ for $C_k(K_h)$,

$$\sigma_i^*(\sigma_j) = \delta_{ij}$$

defines a dual basis $\{\sigma_i^*\}$for $C^k(K_h)$, i.e. given $\omega_h = \sum B_i\sigma_i^*$,

$$\sum B_j\sigma_j^*(\gamma) = \sum B_i A_i = \omega_h^T\gamma.$$

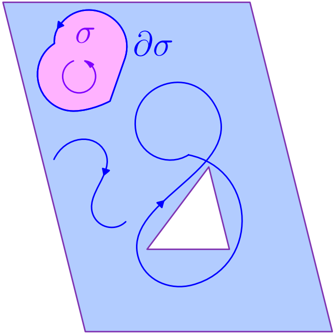
Discrete calculus on primal mesh

- Differential k-forms are naturally integrated over k-chains:

$$\langle R_h \omega, \gamma \rangle = \int_\gamma \omega$$

where $R_h : \Lambda^k \longrightarrow C^k$ defines the operator called the deRham map.

- The boundary $\partial\sigma$ of a k-chain is a $(k-1)$-chain.



- The discrete exterior derivative $\mathrm{d}_h : C^k(K_h) \longrightarrow C^{k+1}(K_h)$ is defined through

$$\langle \mathrm{d}_h \omega_h, \sigma \rangle = \langle \omega_h, \partial\sigma \rangle$$

which guarantes DEC compatibility with Stokes theorem. It is easy to show that

$$R_h \mathrm{d} = \mathrm{d}_h R_h$$

Comparison with other discretizations

- Given an inner product $\langle\cdot,\cdot\rangle_h$ on C^k, we can define the discrete coderivative $\delta_h : C^k \to C^{k-1}$ by enforcing

$$\langle\delta_h\alpha,\beta\rangle_h = \langle\alpha,\mathrm{d}_h\beta\rangle_h.$$

- In mimetic finite differences and in (some interpretation of) finite elements, $\langle\cdot,\cdot\rangle_h$ is naturally induced by a reconstruction operator

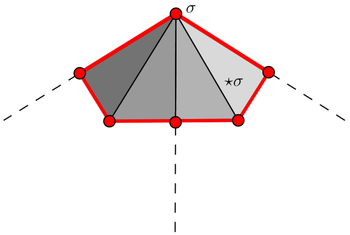
$$W_h : C^k \to L^1\Lambda^k$$

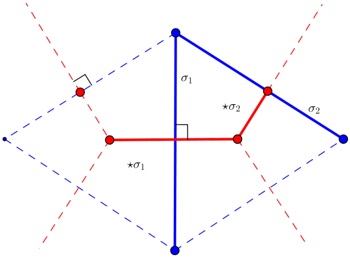
  In other words, cochains are considered as sitting inside a space of continuous forms.

- This is NOT the approach we follow in DEC. The codifferential is built piece by piece (in fact through discretizing the Hodge map).

Circumcentricity and dual cells

- We assume the primal mesh to be completely circumcentric.
- To each $\sigma \in C_k$ is assigned an $(n-k)$-cell $*\sigma$, inducing a dual cell complex $*K_h$.
- The circumcentric dual $*K_h$ is defined on the circumcenters.





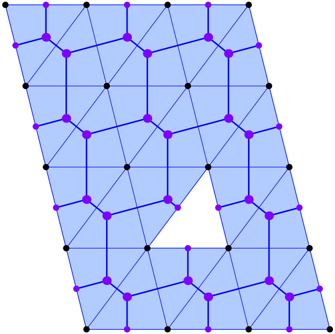
Dual mesh and the codifferential

- Define discrete Hodge star $\star_h : C^k(K_h) \to C^{n-k}(*K_h)$ by

$$\langle \star_h \omega_h, *\sigma \rangle = \frac{|*\sigma|}{|\sigma|} \langle \omega_h, \sigma \rangle.$$

- Discrete codifferential is defined as

$$\delta_h = (-1)^{n(k-1)+1} \star_h \mathrm{d}_h \star_h .$$



- The boundary operator is extended to $*K_h$ by

$$\partial * \tau = (-1)^{k+1} \sum_{\eta \succ \tau} *\eta,$$

where $\eta \in C_{k+1}(K_h)$ is appropriately oriented.

Hilbert space structure

The spaces $C^k(K_h)$ and $C^k(*K_h)$ are finite dimensional Hilbert spaces when respectively equipped with the discrete inner products

$$\left(\alpha_h, \beta_h\right)_h = \sum_{\tau \in \Delta_k(K_h)} \frac{|*\tau|}{|\tau|} \langle \alpha_h, \tau \rangle \langle \beta_h, \tau \rangle = \sum_{\tau \in \Delta_k(K_h)} \langle \alpha_h, \tau \rangle \langle \star_h \beta_h, *\tau \rangle$$

and

$$\left(\star \alpha_h, \star \beta_h\right)_h = \left(\alpha_h, \beta_h\right)_h$$

The Laplace operator and the Poisson problem

The DEC Hodge-Laplacian is finally obtained as

$$\Delta_h : C^k(K_h) \longrightarrow C^k(K_h),$$

$$\Delta_h = \delta_h \mathrm{d}_h + \mathrm{d}_h \delta_h = \pm \star_h \mathrm{d}_h \star_h \mathrm{d}_h \pm \mathrm{d}_h \star_h \mathrm{d}_h \star_h$$

We consider the Poisson problem of finding ω_h

$$\begin{cases} \Delta_h \omega_h = R_h f & \text{in } K_h, \\ \omega_h = R_h g & \text{on } \partial K_h, \end{cases}$$

where f and g are differential forms, and R_h is the deRham operator.

Previous convergence results

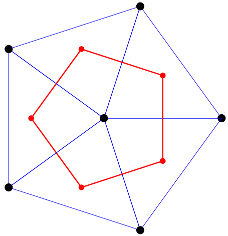
- For p fixed in a shrinking n-gon. Numerical experiments by Xu (2004) revealed

$$(\Delta_h u)(p) - \Delta u(p) \not\to 0 \quad \text{as} \quad h \to 0$$

in general, but

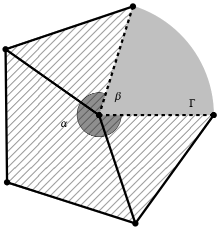
$$(\Delta_h u)(p) - \Delta u(p) = O(h^2)$$

under a very special symmetry assumption.

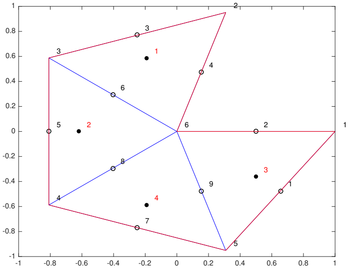


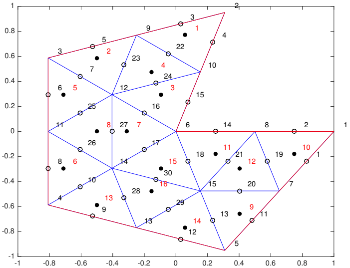
- On the other hand, Nong (2004) observed that $\|\omega - \omega_h\| = O(h^2)$.
- For $n = 2$ and $k = 0$, the matrix $\mathrm{d}_h \star_h \mathrm{d}_h$ is identical to a FEM stiffness matrix (Hildebrandt, Polthier, and Wardetzky 2006).

Numerical experiments 2D



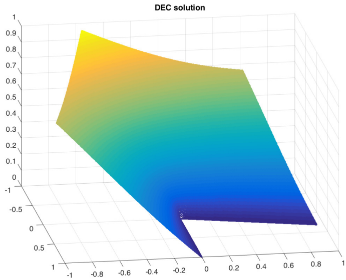
- Model of exact solutions
- Initial primal mesh
- Example of mesh refinement

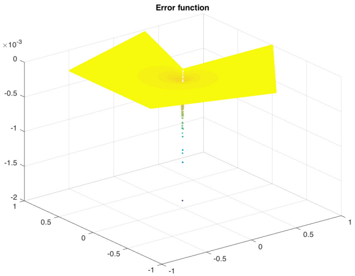




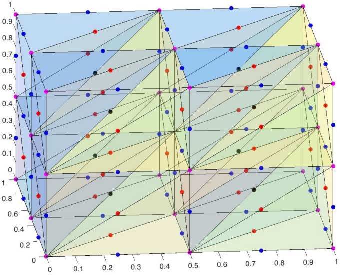
i	$e_i^{\infty} = \|e_{C\cdot 2^{-i}}\|_{\infty}$	$\log\left(e_i^{\infty}/e_{i-1}^{\infty}\right)$	$e^{H_d^1} = \|\mathrm{d}e_{C\cdot 2^{-i}}\|_{C\cdot 2^{-i}}$	$\log\left(e_i^{H_d^1}/e_{i-1}^{H_d^1}\right)$	$e^{L_d^2} = \|e_{C\cdot 2^{-i}}\|_{C\cdot 2^{-i}}$	$\log\left(e_i^{L_d^2}/e_{i-1}^{L_d^2}\right)$
0	0	-	0	-	0	-
1	3.402738e-02	-	8.467970e-02	-	2.346479e-02	-
2	3.194032e-02	9.131748e-02	6.533106e-02	3.742472e-01	1.353817e-02	7.934654e-01
3	2.346298e-02	4.449927e-01	4.496497e-02	5.389676e-01	6.570546e-03	1.042947e+00
4	1.595752e-02	5.561491e-01	2.983035e-02	5.920204e-01	2.970932e-03	1.145097e+00
5	1.054876e-02	5.971636e-01	1.952228e-02	6.116590e-01	1.299255e-03	1.193231e+00
6	6.894829e-03	6.134867e-01	1.270715e-02	6.194814e-01	5.584503e-04	1.218184e+00
7	4.485666e-03	6.201927e-01	8.252738e-03	6.226958e-01	2.377754e-04	1.231830e+00
8	2.912660e-03	6.229847e-01	5.354822e-03	6.240341e-01	1.007013e-04	1.239517e+00

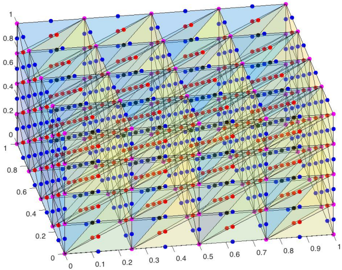
Table: Experiment with $\omega(r,\theta) = r^{\mu}\sin(\mu\theta)$, $\mu = \pi/(2\pi - \beta) = \pi/\alpha = 5/8$.





Numerical experiments 3D





i	$e_i^{\infty} = \|e_{C\cdot 2^{-i}}\|_{\infty}$	$\log\left(e_i^{\infty}/e_{i-1}^{\infty}\right)$	$e^{H_d^1} = \|\mathrm{d}e_{C\cdot 2^{-i}}\|_{C\cdot 2^{-i}}$	$\log\left(e_i^{H_d^1}/e_{i-1}^{H_d^1}\right)$	$e^{L_d^2} = \|e_{C\cdot 2^{-i}}\|_{C\cdot 2^{-i}}$	$\log\left(e_i^{L_d^2}/e_{i-1}^{L_d^2}\right)$
0	8.586493e-04	-	1.487224e-03	-	3.035784e-04	-
1	2.666725e-04	1.687000e+00	6.216886e-04	1.258358e+00	1.156983e-04	1.391702e+00
2	7.122948e-05	1.904523e+00	1.774812e-04	1.808526e+00	3.166206e-05	1.869540e+00
3	1.835021e-05	1.956678e+00	4.594339e-05	1.949737e+00	8.083333e-06	1.969733e+00
4	4.621759e-06	1.989283e+00	1.158904e-05	1.987096e+00	2.031176e-06	1.992635e+00

Table: Experiment with $\omega(x,y) = x^2\sin(y) + \cos(z)$.

Variational crime?

[Holst, Stern '12] Let $i_h : C^*(K_h) \to L^2\Omega(M)$ be a morphism of Hilbert complexes, and let $V_h = i_h C^*(K_h)$. Then

$$\left\| \mathrm{d}(\omega - i_h \omega_h) \right\|_{L^2} \lesssim \mathrm{dist}(\omega, V_h) + \left\| i_h^* i_h - \mathrm{id} \right\|_{C^*(K_h) \to C^*(K_h)}$$

$$\left\langle (i_h^* i_h - \mathrm{id}) u_h, v_h \right\rangle_h = \langle i_h u_h, i_h v_h \rangle - \langle u_h, v_h \rangle_h$$

Take $i_h = W_h$, the Whitney map. Then we can write

$$\langle i_h u_h, i_h v_h \rangle - \langle u_h, v_h \rangle_h = u_h^T (M_h - \star_h) v_h$$

where M_h is the mass matrix, and $\star_h$ is the Hodge matrix.

- For $k = n$, we have $M_h = \star_h$.
- For $k = 0$ and $n = 1$, we have $M_h = \mathrm{tridiag}(h/6, 2h/3, h/6)$ and $\star_h = hI$. So $M_h - \star_h = \mathrm{tridiag}(h/6, -h/3, h/6)$, and

$$\| i_h^* i_h - \mathrm{id} \| \not\to 0.$$

Our strategy

Suppose $\Delta_h \omega_h = R_h f$ and $\Delta\omega = f$, and write $e_h = \omega_h - R_h\omega$ for the error.

- We use a Lax-Richtmyer type of argument, i.e.

$$\begin{aligned}\|e_h\| &\le \|\Delta_h^{-1}\|\,\|\underbrace{\Delta_h(\omega_h - R_h\omega)}_{\text{discrete residual}}\| \\ &= \|\Delta_h^{-1}\|\,\|\Delta_h\omega_h - R_h f + R_h\Delta\omega - \Delta_h R_h\omega\| \\ &= \underbrace{\|\Delta_h^{-1}\|}_{\text{stability}}\,\underbrace{\|R_h\Delta\omega - \Delta_h R_h\omega\|}_{\text{consistency}}\end{aligned}$$

  but a naive application only gives an $O(1)$ bound on the error.
- To obtain convergence, we exploit a special structure of the error.

Reformulating the consistency problem

Lemma

Given $\omega \in C^2\Lambda^k(M)$, we have

$$\Delta_h R_h \omega - R_h \Delta \omega = \star_h \mathrm{d}_h (\star_h R_h - R_h \star) \mathrm{d}\omega + (\star_h R_h - R_h \star) d \star d\omega$$
$$+ \mathrm{d}_h (\star_h R_h - R_h \star) d \star \omega + \mathrm{d}_h \star_h \mathrm{d}_h (\star_h R_h - R_h \star) \omega.$$

Proof (case $k = 0$).

Since $\mathrm{d}_h R_h = R_h \mathrm{d}$, we have

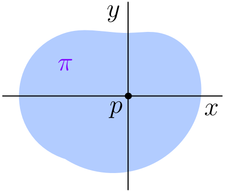
$$\boxed{\star_h \mathrm{d}_h R_h} - R_h \star \mathrm{d} = \star_h R_h \mathrm{d} - R_h \star \mathrm{d} = (\star_h R_h - R_h \star)\mathrm{d}.$$

Therefore

$$\star_h \mathrm{d}_h \boxed{\star_h \mathrm{d}_h R_h} - R_h \star \mathrm{d} \star \mathrm{d} = \star_h \mathrm{d}_h R_h \star \mathrm{d} + \star_h \mathrm{d}_h (\star_h R_h - R_h \star)\mathrm{d} - R_h \star \mathrm{d} \star \mathrm{d}$$
$$= \star_h \mathrm{d}_h (\star_h R_h - R_h \star)\mathrm{d} + (\star_h R_h - R_h \star)\mathrm{d} \star \mathrm{d}.$$

□

Hodge star on 0-cochains



Example

Let $\pi = *p$. For $f \in \Lambda^0(\mathbb{R}^2)$ differentiable,

$$\star f = f \mathrm{d}x \wedge \mathrm{d}y.$$

While $\langle R_h f, p\rangle = f(p)$, we have

$$\begin{aligned}
\langle R_h \star f, \pi\rangle &= \iint_\pi f \mathrm{d}A \\
&= \iint_\pi f(p) + ((x,y) - (p_1, p_2))^T Df(p) + O(h^2) \mathrm{d}A \\
&= |\pi| f(p) + O(h^3) \ \ \big(O(h^4) \text{ if } \pi \text{ is symmetric w.r.t. } p\big).
\end{aligned}$$

We conclude that

$$\langle R_h \star f, \pi\rangle - \langle \star_h R_h f, \pi\rangle = \langle R_h \star f, \pi\rangle - |\pi| f(p) = O(h^3).$$

Hodge star on 1-cochains

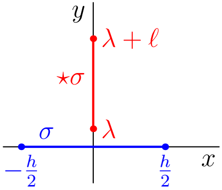
For a 1-form $\omega = f\,\mathrm{d}x + g\,\mathrm{d}y$, we have $\star\omega = f\,\mathrm{d}y - g\,\mathrm{d}x$.

Let $h = |\sigma|$ and $\ell = |*\sigma|$. Then

$$\langle R_h\omega, \sigma\rangle = \int_{-h/2}^{h/2} f\,\mathrm{d}x = hf(0) + O(h^3)$$

and

$$\langle R_h \star \omega, *\sigma\rangle = \int_{\lambda}^{\lambda+\ell} f\,\mathrm{d}y = \ell f(0) + O(\ell^2).$$



We find that

$$\langle R_h \star \omega, *\sigma\rangle = \underbrace{\frac{\ell}{h}\langle R_h\omega, \sigma\rangle}_{\langle \star_h R_h\omega, *\sigma\rangle} + O(\ell^2) + O(\ell h^2).$$

In n-dimensions, we have

$$\star_h R_h\omega - R_h \star \omega = \begin{cases} O(h^n) & \text{in general} \\ O(h^{n+1}) & \text{if } *\sigma \text{ is symmetric wrt } \sigma \end{cases}.$$

Consistency of the discrete Hodge star

Theorem

Let σ be a n-simplex, and suppose $\tau \prec \sigma$ is k-dimensional. Then

$$\langle \star_h R_h \omega, *\tau \rangle = \langle R_h \star \omega, *\tau \rangle + O\left(h^{n+1}/(\gamma_\tau)^k\right), \;\; \omega \in C^1\Lambda^k(\sigma).$$

Corollary

For $\omega \in C^1\Lambda^k(M)$, the estimates

$$\| \star_h R_h \omega - R_h \star \omega \|_\infty = O(h^{n-k+1})$$

and

$$\| \star_h R_h \omega - R_h \star \omega \|_h = O(h)$$

hold when K_h is regular.

Consistency of the discrete Laplacian

If K_h is regular, then

$$\begin{aligned}\Delta_h R_h \omega - R_h \Delta\omega &= \star_h \overbrace{\mathrm{d}_h}^{h^{-1}} \overbrace{(\star_h R_h - R_h \star)}^{h} \mathrm{d}\omega + (\star_h R_h - R_h \star)\, \mathrm{d} \star \mathrm{d}\omega \\ &= O(1) + O(h),\end{aligned}$$

for $\omega \in C^2\Lambda^0(M)$, in both the maximum and discrete L^2-norm.

Integration by parts

Lemma

The discrete codifferential is adjoint to the discrete exteriror derivative, i.e. if $\omega_h \in C^k(K_h)$ and $\eta_h \in C^{k+1}(K_h)$, then $(\mathrm{d}_h \omega_h, \eta_h)_h = (\omega_h, \delta_h \eta_h)_h$.

Proof.

On the one hand,

$$(\mathrm{d}_h \tau^*, \eta_h)_h = \sum_{\sigma} \langle \tau^*, \partial\sigma \rangle \langle \star_h \eta_h, *\sigma \rangle = \langle \tau^*, \tau \rangle \sum_{\sigma \succ \tau} \langle \star \eta_h, *\sigma \rangle,$$

where σ is a $(k+1)$-simplex oriented so that it is consistent with the induced orientation on τ. OTOH, from $\star_h \star_h = (-1)^{k(n-k)}$ on C^k follows $\delta_h = (-1)^k \star_h^{-1} \mathrm{d}_h \star_h$, so

$$(\tau^*, \delta_h \eta_h)_h = (-1)^{k+1} \langle \tau^*, \tau \rangle \langle \mathrm{d}_h \star_h \eta_h, *\tau \rangle = \langle \tau^*, \tau \rangle \sum_{\sigma \succ \tau} \langle \star \eta_h, *\sigma \rangle,$$

where σ is similarly oriented. □

Variational formulation

We compute

$$\left(\delta_h \mathrm{d}_h \omega_h, p^*\right)_h - \left(R_h f, p^*\right)_h = p^*(p) | * p | \left(\langle \delta_h \mathrm{d}_h \omega_h, p \rangle - \langle R_h f, p \rangle\right).$$

In other words,

$$\Delta_h \omega_h = R_h f \iff (\delta_h d_h \omega_h, \nu_h)_h = \left(R_h f, \nu_h\right)_h$$

for all $\nu_h \in C^0(K_h)$.

The homogeneous Poisson problem is thus equivalent to the one of finding $\omega_h \in C^0(K_h)$ with $\omega_h|_{\partial K_h} \equiv 0$ such that

$$(\mathrm{d}_h \omega_h, \mathrm{d}_h \nu_h)_h = \left(R_h f, \nu_h\right)_h \quad \forall \nu_h \in C^0 \cap \{\nu_h|_{\partial K_h} \equiv 0\}.$$

Existence and uniqueness

For $u_h \in C^0$ in general,

$$(\mathrm{d}_h u_h, \mathrm{d}_h u_h)_h = 0 \iff u_h = \text{constant}.$$

We conclude that $\Delta_h = \delta_h \mathrm{d}_h$ is invertible over $\{v_h|_{\partial K_h} \equiv 0\}$, and deduce the existence and uniqueness of discrete solutions.

Equivalence of norms

Linearly extending $W_h\omega_h(\tau) = \sum_\tau \omega_h(\tau)\phi_\tau$, where

$$\phi_\tau = k! \sum_{i=0}^{k} (-1)^i \lambda_i \mathrm{d}\lambda_1 \wedge \ldots \wedge \widehat{\mathrm{d}\lambda_i} \wedge \ldots \wedge \mathrm{d}\lambda_k,$$

and λ_i is the piecewise linear hat function on the ith vertex of τ, defines the Whitney map from the space of cochains to the Whitney forms.

Theorem

Let K_h be a family of regular triangulations. There exist two positive constants c_1 and c_2, independent of h, satisfying

$$c_1\|\omega_h\|_h \leq \|W_h\omega_h\|_{L^2\Lambda^k(K_h)} \leq c_2\|\omega_h\|_h, \;\; \omega_h \in C^k(K_h).$$

Discrete Poincaré inequality on 0-cochains

Corollary

There exists a constant C, independent of h, such that the discrete Poincare inequality

$$\|\omega_h\|_h \leq C\|\mathrm{d}_h\omega_h\|_h$$

holds for all $\omega_h \in C^0(K_h)$ such that $\omega_h = 0$ on ∂K_h.

Proof.

Using the previous theorem and the Poincare inequality, we have

$$\|\omega_h\|_h \lesssim \|W_h\omega_h\|_{L^2\Lambda^k(K_h)} \lesssim \|\mathrm{d}W_h\omega_h\|_{L^2\Lambda^k(K_h)} = \|W_h\mathrm{d}_h\omega_h\|_{L^2\Lambda^k(K_h)} \lesssim \|\mathrm{d}_h\omega_h\|_h.$$

□

Stability

We have

$$(\mathrm{d}_h\omega_h, \mathrm{d}_h\omega_h)_h = \left(R_h f, \omega_h\right)_h \le \|R_h f\|_h \|\omega_h\|_h \le C\|R_h f\|_h \|\mathrm{d}_h\omega_h\|_h$$

Hence

$$\|\omega_h\|_h \le C\|R_h f\|_h \qquad \text{i.e.,} \qquad \|\Delta_h^{-1}\| \le C$$

Coupled with

$$\Delta_h R_h \omega - R_h \Delta\omega = O(1)$$

this only gives

$$\|e_h\|_h \le \|\Delta_h^{-1}\| \cdot \|\Delta_h R_h \omega - R_h \Delta\omega\|_h = O(1)$$

Convergence in L_2

Our consistency and stability estimates only yields $\|e_h\|_h = O(1)$.

However,

$$\begin{aligned}
(\mathrm{d}_h e_h, \mathrm{d}_h e_h)_h &= (\Delta_h e_h, e_h)_h \\
&= (\star_h \mathrm{d}_h (\star_h R_h - R_h \star)\,\mathrm{d}\omega, e_h)_h + ((\star_h R_h - R_h \star)\,\mathrm{d}\star\mathrm{d}\omega, e_h)_h \\
&= \left(\star_h^{-1}(\star_h R_h - R_h \star)\,\mathrm{d}\omega, \mathrm{d}_h e_h\right)_h + ((\star_h R_h - R_h \star)\,\mathrm{d}\star\mathrm{d}\omega, e_h)_h \\
&\le Ch\|\mathrm{d}_h e\|_h + Ch\|e_h\|_h.
\end{aligned}$$

Theorem

The discrete solutions $\omega_h \in C^0(K_h)$ of the Dirichlet Poisson problem for 0-forms over a regular triangulation K_h satisfy

$$\boxed{\|e_h\|_h \le C\|\mathrm{d}_h e_h\|_h = O(h)}$$

Open problems and references

Open problems

- Higher degree forms
- Duality argument?
- Convergence in uniform norm
- Numerical experiments
- Eigenvalue problems

References

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