Counting Quasiplatonic Cyclic Group Actions of Order n

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Main Question

How many compact Riemann surfaces X admit a conformal cyclic group action of order n, if we assume $X \cong \mathbb{H}/\Gamma$ with $\Gamma \lhd \Delta(n_1, n_2, n_3)$ and $\Delta(n_1, n_2, n_3)/\Gamma \cong C_n$?

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These surfaces are called **quasiplatonic cyclic n-gonal surfaces**.

Related Question

We will apply results of Benim and Wootton to count all **topological cyclic group actions of order n on quasiplatonic surfaces** (this is different from counting n-gonal surfaces).

Group Acting on a Surface

A group G **acts topologically** on a surface X of genus $g \geq 2$ if there is a monomorphism $\epsilon : G \to \mathrm{Homeo}^+(X)$.

Two actions ϵ_1 and ϵ_2 are **equivalent** if $\epsilon_1(G)$ and $\epsilon_2(G)$ are conjugate in $\mathrm{Homeo}^+(X)$.

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The number $R(C_n)$ of regular cyclic dessins of order $n \geq 7$ having genus at least two is given by

$$R(C_n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right) - 3.$$

(G. Jones, 2014)

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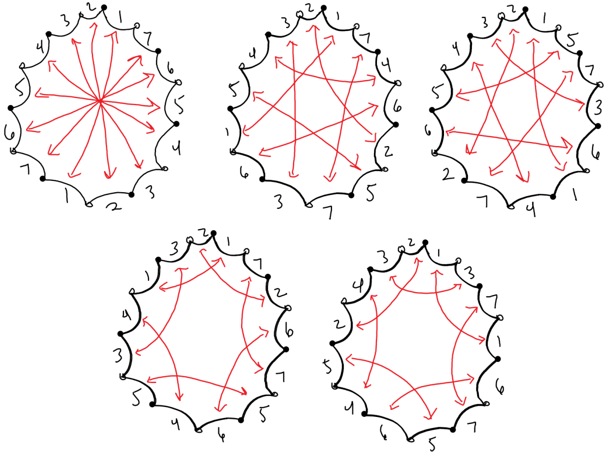
1. $y^2 = x^8 - x$,
2. $y^7 = x(x-1)^2$ (Klein's Quartic).

Example

There are **five** regular cyclic dessins on quasiplatonic cyclic 7-gonal surfaces.



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1. Is there a closed form for $QC(n)$?
2. What is the relationship between $QC(n)$ and $R(C_n)$?
3. Can $QC(n)$ be determined combinatorially, by using dessins for instance?

Method - Harvey's Theorem for the Quasiplatonic Case

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Theorem (Harvey, 1966)

Let $n = lcm(n_1, n_2, n_3)$. Then the cyclic group of order n acts on X of genus g with signature (n_1, n_2, n_3) if and only if

1. $n = lcm(n_1, n_2) = lcm(n_1, n_3) = lcm(n_2, n_3)$;
2. *for n even, exactly two of n_1, n_2, n_3 must be divisible by the maximum power of two dividing n;*
3. *the Riemann-Hurwitz formula holds:*

$$g = 1 + \frac{n}{2}\left(1 - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}\right).$$

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Fix an equivalence class of (n_1, n_2, n_3)-generating vectors for C_n. This determines a triangle group $\Delta(n_1, n_2, n_3)$ and a torsion-free Fuchsian group Γ with $\Delta(n_1, n_2, n_3)/\Gamma \cong C_n$.

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There are three cases for possible signatures (n_1, n_2, n_3):

- all n_i are distinct;
- exactly two of n_i are equal;
- all n_i are equal.

Method - Benim/Wootton Formulas

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Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$ be the prime factorization of n.

Signature	T = number of distinct topological actions
(n_1, n_2, n_3)	$T = \phi(gcd(n_1, n_2, n_3)) \left(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1} \right)$
(n_1, n, n)	$T = \frac{1}{2} \left(\tau_1(n, n_1) + \phi(n) \left(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1} \right) \right)$
(n, n, n)	$T = \frac{1}{6} \left(3 + 2\tau_2(n) + \phi(n) \left(\prod_{i=1}^{r} \frac{p_i - 2}{p_i - 1} \right) \right)$

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Here,

- $\tau_1(n, n_1)$ = number of noncongruent, nonzero solutions to $x^2 + 2x \equiv 0 \bmod n$ where $gcd(x, n) = n/n_1$;
- $\tau_2(n)$ = number of noncongruent solutions to $x^2 + x + 1 \equiv 0 \bmod n$;
- $w \geq 0$ is an integer representing the number of primes (including multiplicity) shared in common.

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1. find all admissible signatures for a given n;
2. for each signature, use one of three different Benim/Wootton formulas giving the number of nonequivalent quasiplatonic cyclic actions on surfaces of that signature;
3. add up all values given by the formulas from all possible signatures for n. This number will be $QC(n)$.

Example

Let $n = 20$.

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Signature	T
$(4, 5, 20)$	$T = 1$
$(4, 10, 20)$	$T = 1$
$(2, 20, 20)$	$T = 1$
$(5, 20, 20)$	$T = 2$
$(10, 20, 20)$	$T = 2$

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Then $QC(20) = 1 + 1 + 1 + 2 + 2 = 7$.

Example

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For $n = p \geq 5$ a prime, there is only one admissible signature: (p, p, p). Then

$$QC(p) = \frac{1}{6}\left(3 + 2\tau_2(p) + \phi(p)\left(\frac{p-2}{p-1}\right)\right)$$
$$= \begin{cases} \frac{1}{6}(p+1) & p \equiv 5 \bmod 6 \\ \frac{1}{6}(p+1) + \frac{2}{3} & p \equiv 1 \bmod 6 \end{cases} .$$

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$QC(n)$ is known for some values of n (e.q., n is a prime power). The general case is still being investigated.

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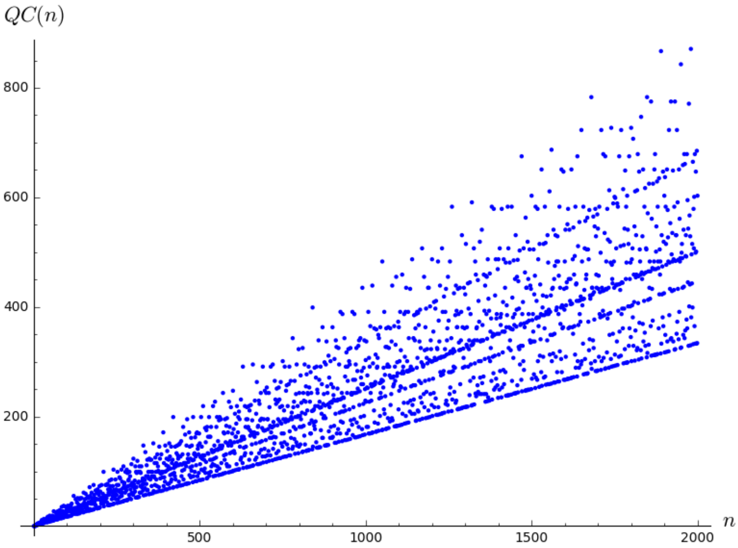
Let $QC_R(n) := 6 \cdot QC(n) - R(C_n)$. Computations with Sage suggest that, for certain families of positive integers, $QC_R(n)$ is a constant.

Data - Table of Values

n	$QC(n)$	$R(C_n)$	$QC_R(n)$
7	2	5	7
8	3	9	9
9	2	9	3
10	3	15	3
11	2	9	3
12	5	21	9
13	3	11	7
14	4	21	3
15	5	21	9
16	5	21	9
17	3	15	3
18	6	33	3
19	4	17	7
20	7	33	9

n	$QC(n)$	$R(C_n)$	$QC_R(n)$
21	7	29	13
22	6	33	3
23	4	21	3
24	11	45	21
25	5	27	3
26	7	39	3
27	6	33	3
28	9	45	9
29	5	27	3
30	13	69	9

Data - Graph of $QC(n)$



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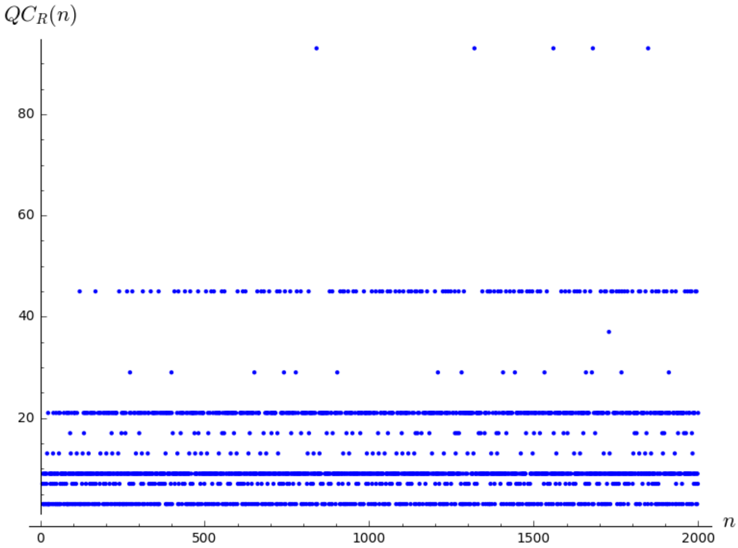

Future Directions

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- Generalize methods to any quasiplatonic group; i.e., find all topological actions of $G = \Delta/\Gamma$ on surfaces $X \cong \mathbb{H}/\Gamma$, for Δ a triangle group and Γ a surface group.

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- Generalize methods to any quasiplatonic group; i.e., find all topological actions of $G = \Delta/\Gamma$ on surfaces $X \cong \mathbb{H}/\Gamma$, for Δ a triangle group and Γ a surface group.
- Compute $QC(n)$ using combinatorial information from the regular cyclic dessins.
- Relate topological actions to conformal actions.

References

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Questions? Thank you!