Min-sum and network flows

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(joint work with Sekhar Tatikonda)

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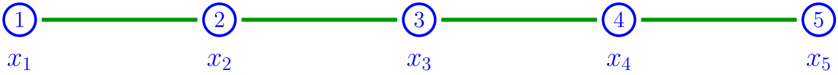
Optimization and Inference for Physical Flows on Networks
Banff International Research Station

March 8, 2017

(NSF Grant: *Locality in Network Optimization*, Award no. 1609484, ECCS)

Min-sum: path graph



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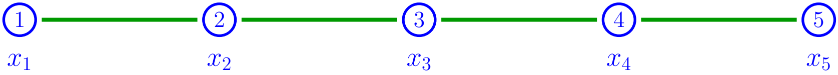
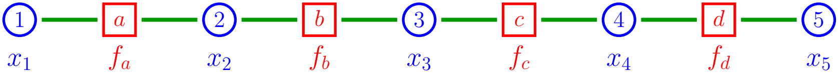


$$\min_{x_1} \min_{x_2} \min_{x_3} \min_{x_4} \min_{x_5} \underbrace{\left(f_a(x_1, x_2) + f_b(x_2, x_3) + f_c(x_3, x_4) + f_d(x_4, x_5) \right)}_{f(x)}$$

Min-sum: path graph



$$\min_{x_1} \min_{x_2} \min_{x_3} \min_{x_4} \min_{x_5} \underbrace{\left(f_a(x_1, x_2) + f_b(x_2, x_3) + f_c(x_3, x_4) + f_d(x_4, x_5) \right)}_{f(x)}$$

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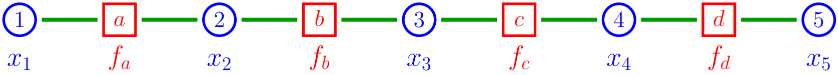
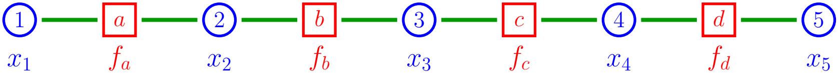


$$\min_{x_1} \min_{x_2} \min_{x_3} \min_{x_4} \min_{x_5} \underbrace{\left(f_a(x_1, x_2) + f_b(x_2, x_3) + f_c(x_3, x_4) + f_d(x_4, x_5) \right)}_{f(x)}$$

$x_i \in \{0, 1\}$ **naive algorithm** $O(2^n)$

Min-sum: path graph



$$\min_{x_1} \min_{x_2} \min_{x_3} \min_{x_4} \boxed{\min_{x_5}} \left(f_a(x_1, x_2) + f_b(x_2, x_3) + f_c(x_3, x_4) + f_d(x_4, x_5) \right)$$

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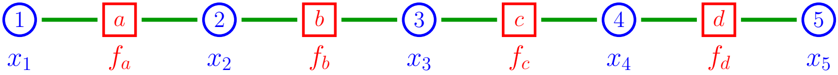

$$\min_{x_1} \min_{x_2} \min_{x_3} \min_{x_4} \left(f_a(x_1, x_2) + f_b(x_2, x_3) + f_c(x_3, x_4) + \min_{x_5} f_d(x_4, x_5) \right)$$

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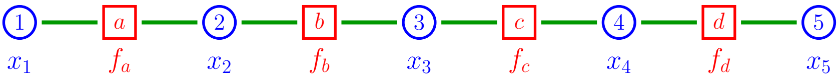

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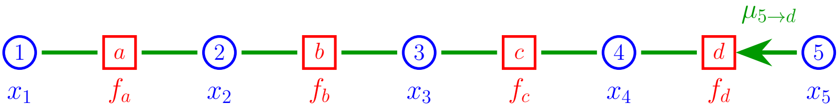

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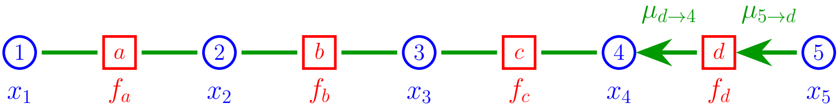
$\mu_{5\to d} = 0$

Min-sum: path graph



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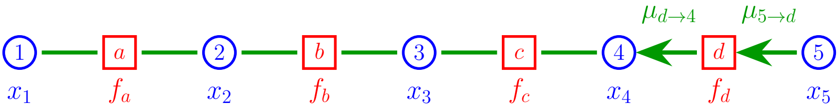

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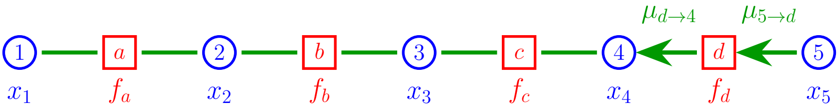

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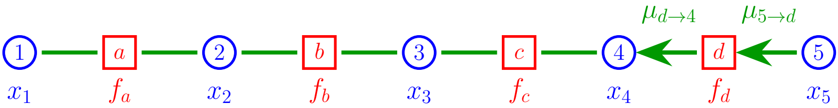

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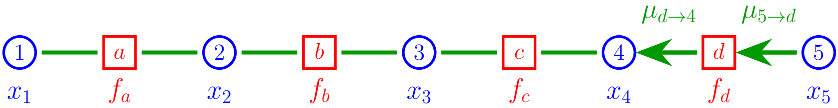

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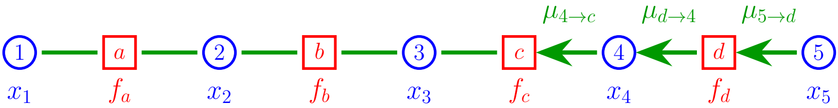

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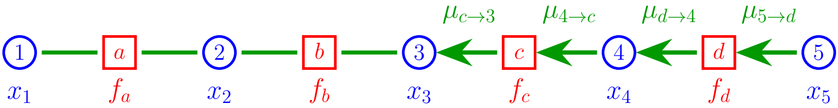

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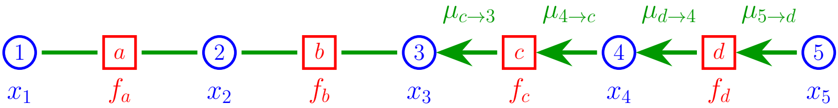

$$\min_{x_1} \min_{x_2} \min_{x_3} \left(f_a(x_1, x_2) + f_b(x_2, x_3) + \mu_{c\to 3}(x_3) \right)$$

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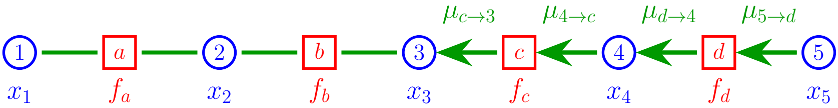

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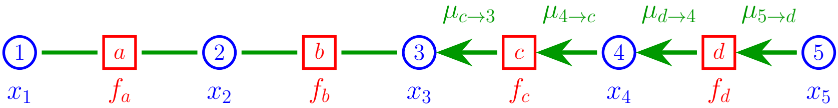

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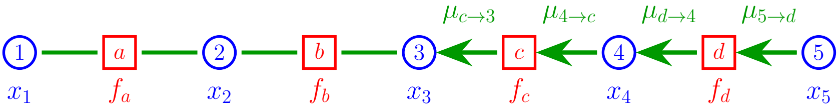

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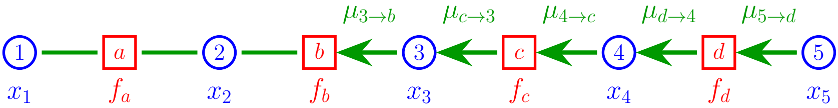

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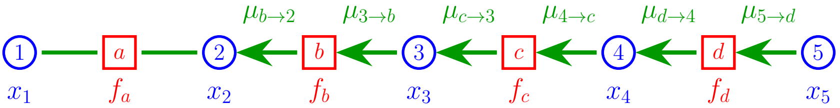

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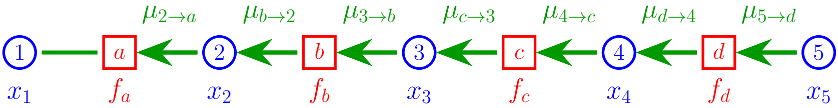

$$\min_{x_1} \min_{x_2} \left(f_a(x_1, x_2) + \mu_{b\to 2}(x_2) \right)$$

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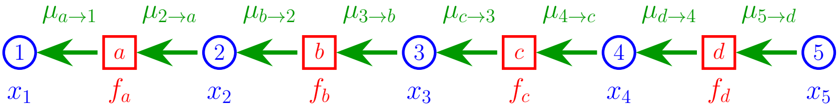

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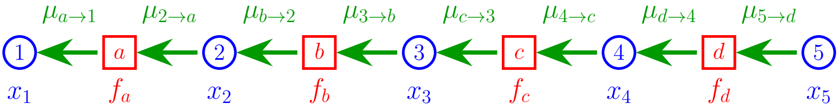

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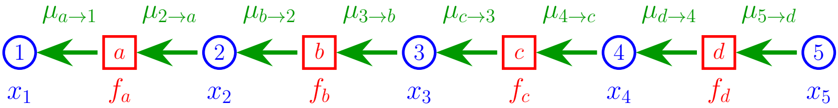

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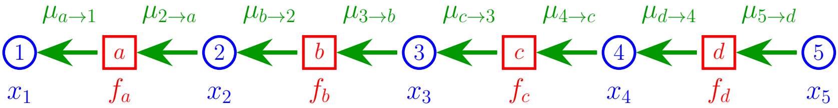
$$\mu_{2\to a} = \mu_{b\to 2}$$

Min-sum: path graph



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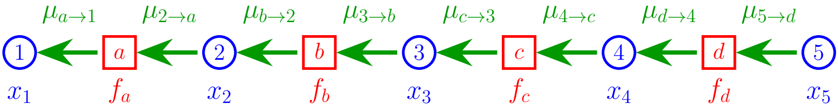
Min-sum: path graph

$O(n)$ **Dynamic Programming**



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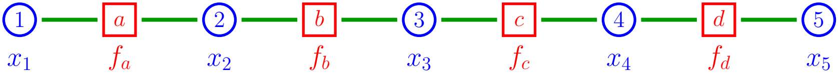
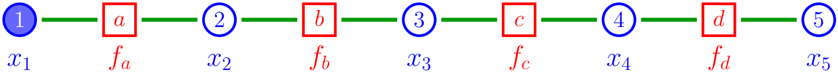
$O(n)$ **Dynamic Programming**



$$\min_{x_1} \min_{x_2} \min_{x_3} \min_{x_4} \min_{x_5} \underbrace{\left(f_a(x_1, x_2) + f_b(x_2, x_3) + f_c(x_3, x_4) + f_d(x_4, x_5) \right)}_{f(x)}$$

Min-sum: path graph

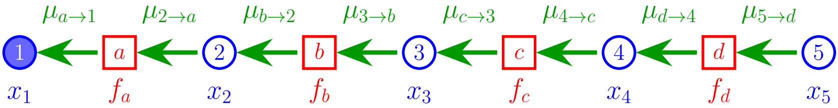
$O(n)$ **Dynamic Programming**



$$\min_{x_1}\min_{x_2}\left(f_a(x_1,x_2)+\min_{x_3}\left(f_b(x_2,x_3)+\min_{x_4}\left(f_c(x_3,x_4)+\min_{x_5}f_d(x_4,x_5)\right)\right)\right)$$

Min-sum: path graph

$O(n)$ **Dynamic Programming**

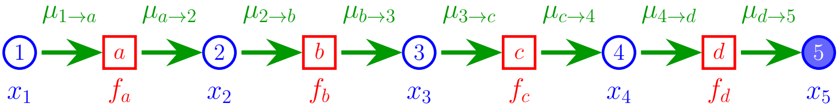


$$\min_{x_1} \underbrace{\min_{x_2}\left(f_a(x_1,x_2) + \underbrace{\min_{x_3}\left(f_b(x_2,x_3) + \underbrace{\min_{x_4}\left(f_c(x_3,x_4) + \underbrace{\min_{x_5} f_d(x_4,x_5)}_{\mu_{d\to 4}(x_4)}\right)}_{\mu_{c\to 3}(x_3)}\right)}_{\mu_{b\to 2}(x_2)}\right)}_{\mu_{a\to 1}(x_1)}$$

$$= \min_{x_1} \mu_{a\to 1}(x_1)$$

Min-sum: path graph

$O(n)$ **Dynamic Programming**

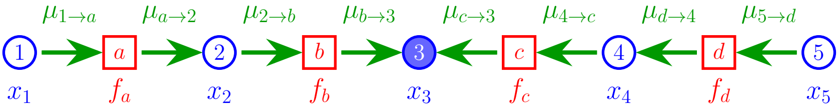


$$\underbrace{\min_{x_5} \min_{x_4} \left(\underbrace{\min_{x_3} \left(\underbrace{\min_{x_2} \left(\underbrace{\min_{x_1} f_a(x_1, x_2)}_{\mu_{a\to 2}(x_2)} + f_b(x_2, x_3) \right)}_{\mu_{b\to 3}(x_3)} + f_c(x_3, x_4) \right)}_{\mu_{c\to 4}(x_4)} + f_d(x_4, x_5) \right)}_{\mu_{d\to 5}(x_5)}$$

$$= \min_{x_5} \mu_{d\to 5}(x_5)$$

Min-sum: path graph

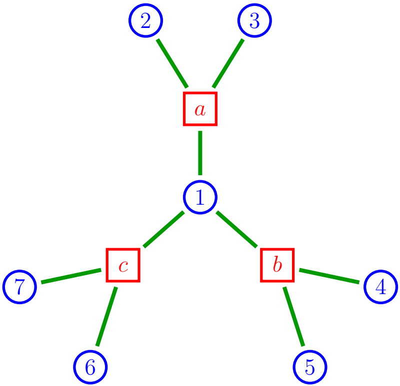
$O(n)$ **Dynamic Programming**



$$\underbrace{\min_{x_3}\left(\min_{x_2}\left(\underbrace{\min_{x_1} f_a(x_1,x_2)}_{\mu_{a\to 2}(x_2)} + f_b(x_2,x_3)\right)}_{\mu_{b\to 3}(x_3)} + \underbrace{\min_{x_4}\left(f_c(x_3,x_4) + \underbrace{\min_{x_5} f_d(x_4,x_5)}_{\mu_{d\to 4}(x_4)}\right)}_{\mu_{c\to 3}(x_3)}\right)$$

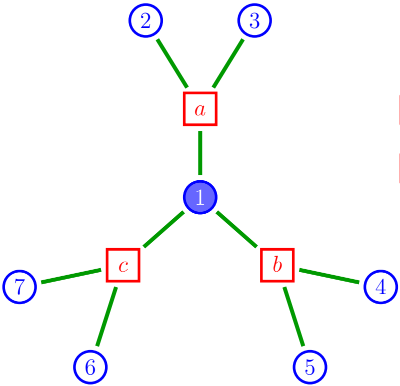
$$= \min_{x_3}\left(\mu_{b\to 3}(x_3) + \mu_{c\to 3}(x_3)\right)$$

Min-sum: trees



$$\min_{x}\left(f_a(x_1, x_2, x_3) + f_b(x_1, x_4, x_5) + f_c(x_1, x_6, x_7)\right)$$

Min-sum: trees



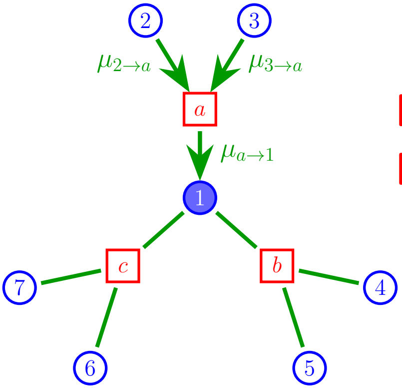
Dynamic Programming

$O(n)$

$$\min_{x}\Big(f_a(x_1,x_2,x_3)+f_b(x_1,x_4,x_5)+f_c(x_1,x_6,x_7)\Big)$$

$$=\min_{x_1}\Big(\min_{x_2,x_3} f_a(x_1,x_2,x_3)+\min_{x_4,x_5} f_b(x_1,x_4,x_5)+\min_{x_6,x_7} f_c(x_1,x_6,x_7)\Big)$$

Min-sum: trees



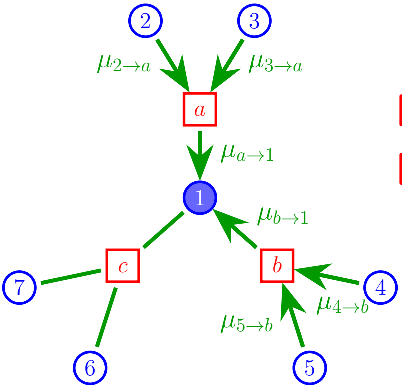
Dynamic Programming

$O(n)$

$$\min_{x}\left(f_a(x_1,x_2,x_3)+f_b(x_1,x_4,x_5)+f_c(x_1,x_6,x_7)\right)$$

$$=\min_{x_1}\left(\underbrace{\min_{x_2,x_3} f_a(x_1,x_2,x_3)}_{\mu_{a\to 1}(x_1)}+\min_{x_4,x_5} f_b(x_1,x_4,x_5)+\min_{x_6,x_7} f_c(x_1,x_6,x_7)\right)$$

Min-sum: trees



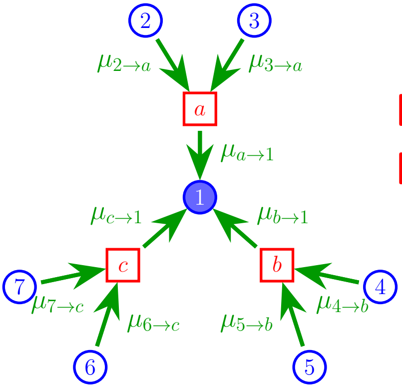
Dynamic Programming

$O(n)$

$$\min_x \left(f_a(x_1, x_2, x_3) + f_b(x_1, x_4, x_5) + f_c(x_1, x_6, x_7) \right)$$

$$= \min_{x_1} \left(\underbrace{\min_{x_2, x_3} f_a(x_1, x_2, x_3)}_{\mu_{a\to 1}(x_1)} + \underbrace{\min_{x_4, x_5} f_b(x_1, x_4, x_5)}_{\mu_{b\to 1}(x_1)} + \min_{x_6, x_7} f_c(x_1, x_6, x_7) \right)$$

Min-sum: trees



Dynamic Programming

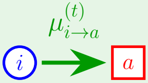
$O(n)$

$$\min_{x}\left(f_a(x_1,x_2,x_3)+f_b(x_1,x_4,x_5)+f_c(x_1,x_6,x_7)\right)$$

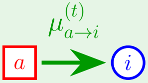
$$=\min_{x_1}\left(\underbrace{\min_{x_2,x_3} f_a(x_1,x_2,x_3)}_{\mu_{a\to 1}(x_1)}+\underbrace{\min_{x_4,x_5} f_b(x_1,x_4,x_5)}_{\mu_{b\to 1}(x_1)}+\underbrace{\min_{x_6,x_7} f_c(x_1,x_6,x_7)}_{\mu_{c\to 1}(x_1)}\right)$$

Messages

variable → factor

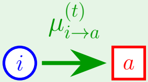


factor → variable



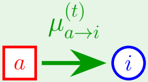
Messages

variable $\rightarrow$ factor



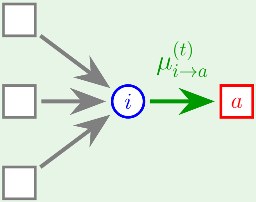
$\mu^{(t)}_{i\to a}(x_i) =$

factor $\rightarrow$ variable



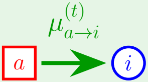
Messages

variable → factor



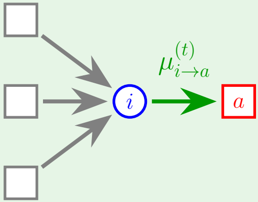
$\mu^{(t)}_{i\to a}(x_i) =$

factor → variable



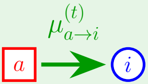
Messages

variable → factor



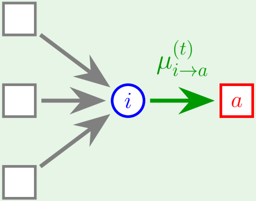
$$\mu^{(t)}_{i\to a}(x_i) = \sum_{b\in\partial i\setminus a} \mu^{(t-1)}_{b\to i}(x_i)$$

factor → variable



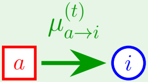
Messages

variable → factor



$$\mu^{(t)}_{i\to a}(x_i) = \sum_{b\in\partial i\setminus a} \mu^{(t-1)}_{b\to i}(x_i)$$

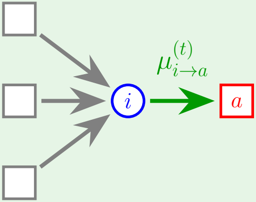
factor → variable



$$\mu^{(t)}_{a\to i}(x_i) =$$

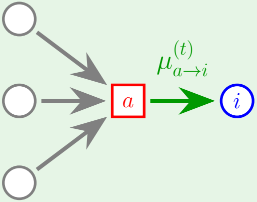
Messages

variable $\to$ factor



$$\mu^{(t)}_{i\to a}(x_i) = \sum_{b\in\partial i\setminus a} \mu^{(t-1)}_{b\to i}(x_i)$$

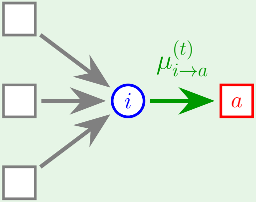
factor $\to$ variable



$$\mu^{(t)}_{a\to i}(x_i) =$$

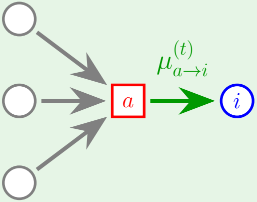
Messages

variable $\rightarrow$ factor



$$\mu_{i\to a}^{(t)}(x_i) = \sum_{b\in\partial i\setminus a} \mu_{b\to i}^{(t-1)}(x_i)$$

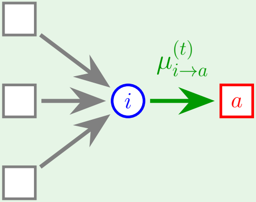
factor $\rightarrow$ variable



$$\mu_{a\to i}^{(t)}(x_i) = \sum_{j\in\partial a\setminus i} \mu_{j\to a}^{(t-1)}(x_j)$$

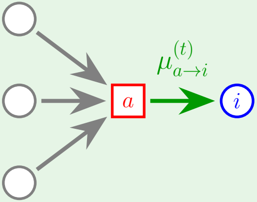
Messages

variable $\rightarrow$ factor



$$\mu_{i\rightarrow a}^{(t)}(x_i) = \sum_{b\in\partial i\setminus a} \mu_{b\rightarrow i}^{(t-1)}(x_i)$$

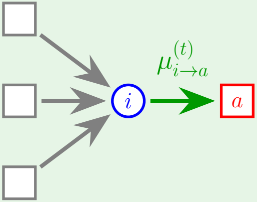
factor $\rightarrow$ variable



$$\mu_{a\rightarrow i}^{(t)}(x_i) = \sum_{j\in\partial a\setminus i} \mu_{j\rightarrow a}^{(t-1)}(x_j) + f_a(x_{\partial a\setminus i}, x_i)$$

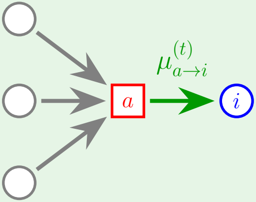
Messages

variable → factor



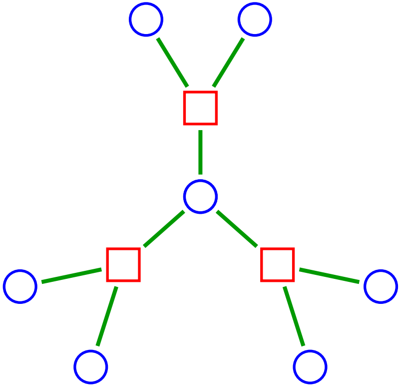
$$\mu_{i\to a}^{(t)}(x_i) = \sum_{b\in\partial i\setminus a} \mu_{b\to i}^{(t-1)}(x_i)$$

factor → variable



$$\mu_{a\to i}^{(t)}(x_i) = \min_{x_{\partial a\setminus i}} \left(\sum_{j\in\partial a\setminus i} \mu_{j\to a}^{(t-1)}(x_j) + f_a(x_{\partial a\setminus i}, x_i) \right)$$

Min-sum: beyond trees?

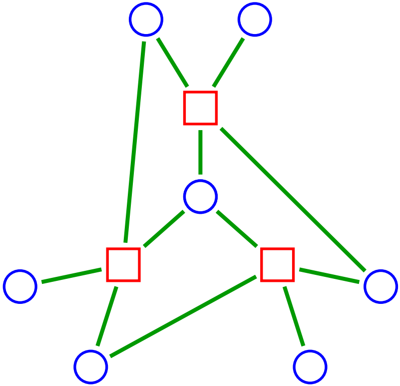


Optimal solution:

$$x^\star := \arg\min_{x} \sum_{a \in \text{factors}} f_a(x_{\partial a})$$

Min-sum: beyond trees?

?

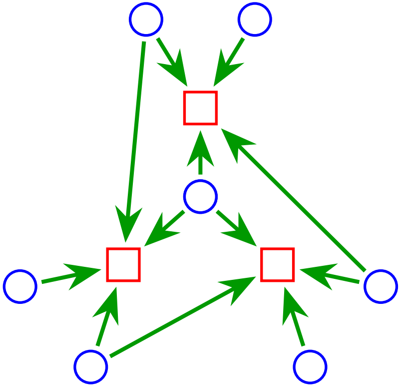


Optimal solution:

$$x^\star := \arg\min_x \sum_{a \in \text{factors}} f_a(x_{\partial a})$$

Min-sum: beyond trees?

time 1: $\{\mu^{(1)}_{i\to a} = 0\}$



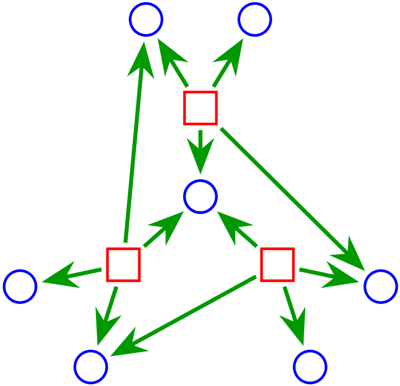
Optimal solution:

$$x^\star := \arg\min_x \sum_{a \in \text{factors}} f_a(x_{\partial a})$$

Min-sum: beyond trees?

time 1: $\{\mu^{(1)}_{i\to a} = 0\}$

time 2: $\{\mu^{(2)}_{a\to i}\}$



Optimal solution:

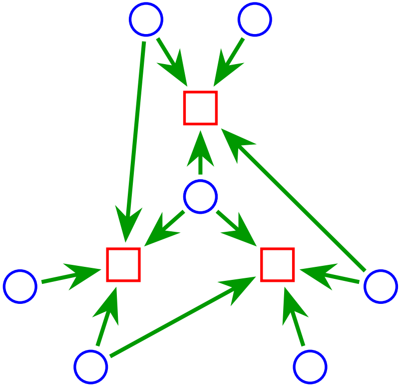
$$x^\star := \arg\min_x \sum_{a \in \text{factors}} f_a(x_{\partial a})$$

Min-sum: beyond trees?

time 1: $\{\mu^{(1)}_{i\to a} = 0\}$

time 2: $\{\mu^{(2)}_{a\to i}\}$

time 3: $\{\mu^{(3)}_{i\to a}\}$



Optimal solution:

$$x^\star := \arg\min_x \sum_{a \in \text{factors}} f_a(x_{\partial a})$$

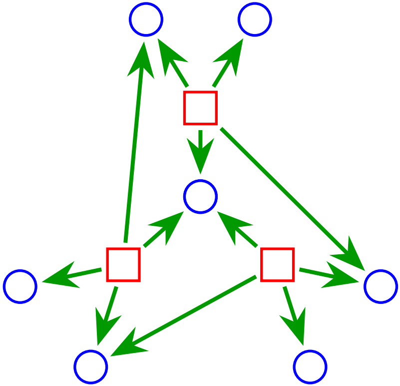
Min-sum: beyond trees?

time 1: $\{\mu^{(1)}_{i\to a} = 0\}$

time 2: $\{\mu^{(2)}_{a\to i}\}$

time 3: $\{\mu^{(3)}_{i\to a}\}$

time 4: $\{\mu^{(4)}_{a\to i}\}$



Optimal solution:

$$x^\star := \arg\min_x \sum_{a\in\text{factors}} f_a(x_{\partial a})$$

Min-sum: beyond trees?

time 1: $\{\mu^{(1)}_{i\to a} = 0\}$

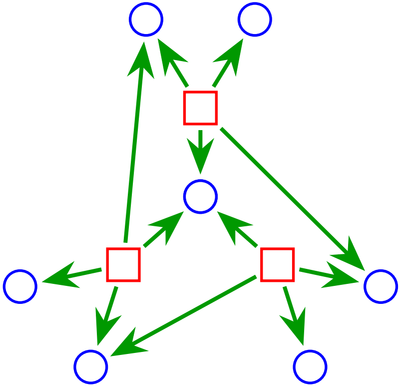
time 2: $\{\mu^{(2)}_{a\to i}\}$

time 3: $\{\mu^{(3)}_{i\to a}\}$

time 4: $\{\mu^{(4)}_{a\to i}\}$

$\vdots$

time t: $\{\mu^{(t)}_{a\to i}\}$



Optimal solution:

$$x^\star := \arg\min_x \sum_{a\in\text{factors}} f_a(x_{\partial a})$$

Min-sum: beyond trees?

time 1: $\{\mu^{(1)}_{i\to a} = 0\}$

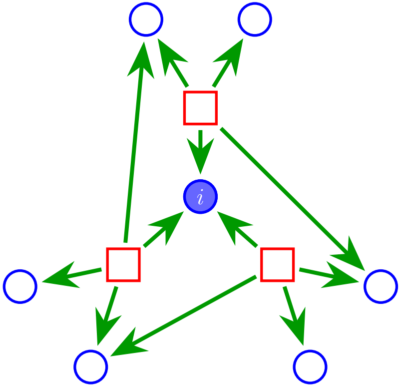
time 2: $\{\mu^{(2)}_{a\to i}\}$

time 3: $\{\mu^{(3)}_{i\to a}\}$

time 4: $\{\mu^{(4)}_{a\to i}\}$

⋮

time t: $\{\mu^{(t)}_{a\to i}\}$



Estimate time t:

$$\hat{x}^{(t)}_i := \arg\min_{x_i} \sum_{a\in\partial i} \mu^{(t)}_{a\to i}(x_i)$$

Optimal solution:

$$x^\star := \arg\min_{x} \sum_{a\in \text{factors}} f_a(x_{\partial a})$$

Min-sum: beyond trees?

time 1: $\{\mu_{i \to a}^{(1)} = 0\}$

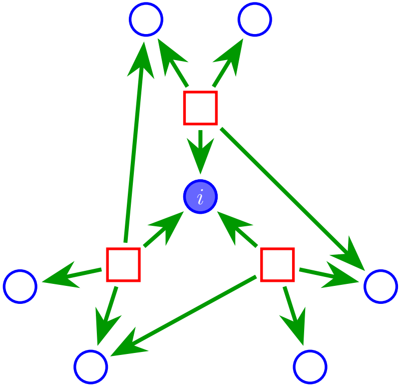
time 2: $\{\mu_{a \to i}^{(2)}\}$

time 3: $\{\mu_{i \to a}^{(3)}\}$

time 4: $\{\mu_{a \to i}^{(4)}\}$

⋮

time t: $\{\mu_{a \to i}^{(t)}\}$



?

Estimate time t:

$$\hat{x}_i^{(t)} := \arg\min_{x_i} \sum_{a \in \partial i} \mu_{a \to i}^{(t)}(x_i)$$

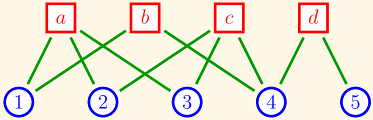
→

Optimal solution:

$$x^\star := \arg\min_{x} \sum_{a \in \text{factors}} f_a(x_{\partial a})$$

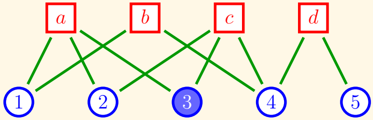
Computation tree

Graph



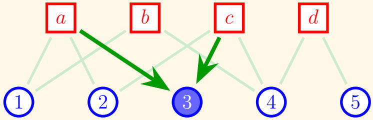
Computation tree

Graph



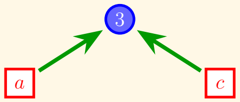
Computation tree

Graph

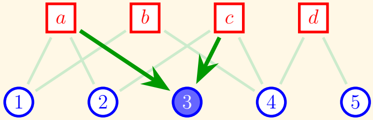


Computation tree

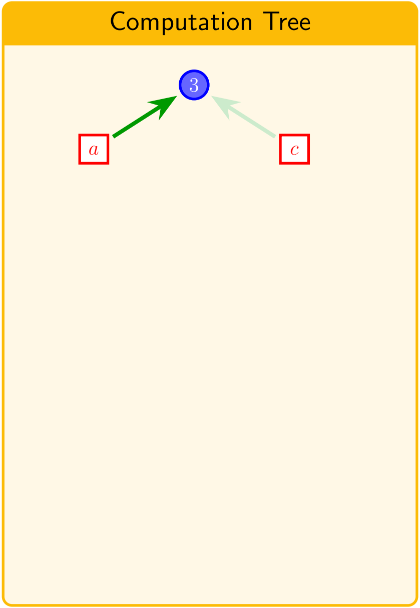
Computation Tree

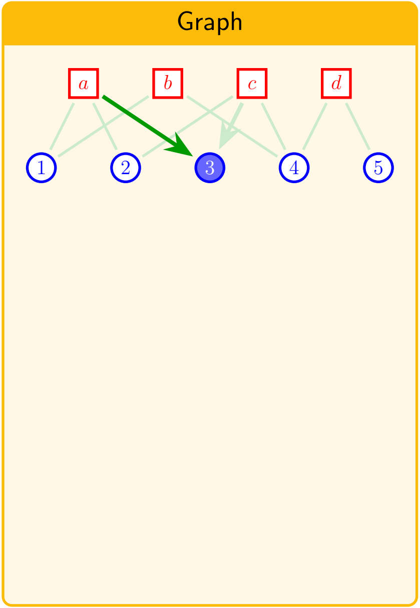


Graph



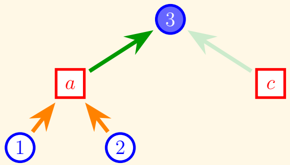
Computation tree



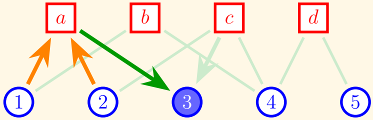


Computation tree

Computation Tree

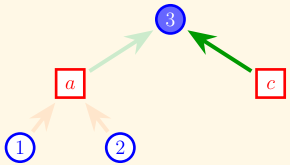


Graph

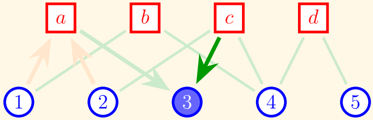


Computation tree

Computation Tree



Graph



Computation tree

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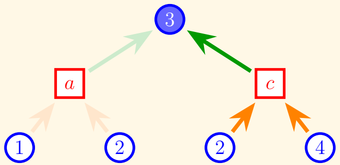
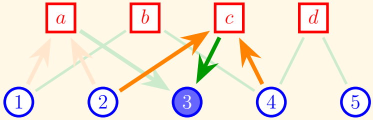


Graph



Computation tree

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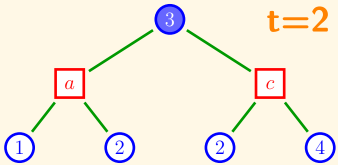
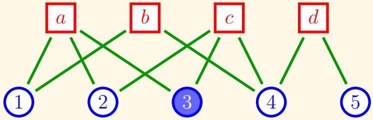


Graph



Computation tree

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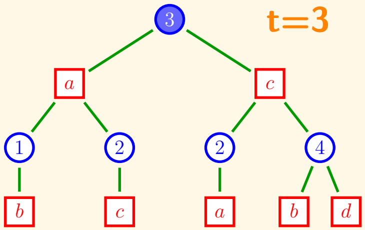
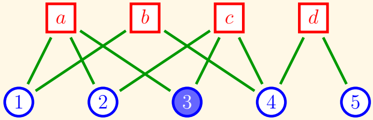


Graph



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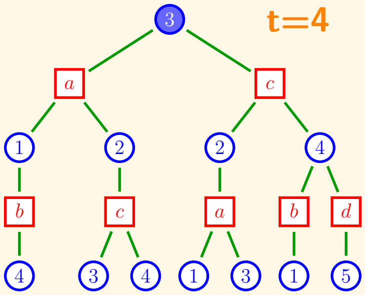
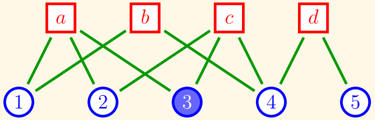


Graph



Computation tree

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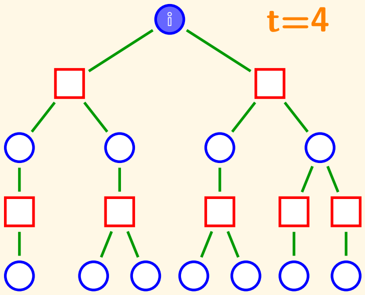
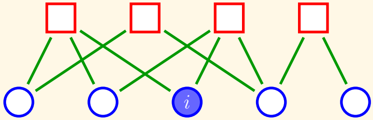
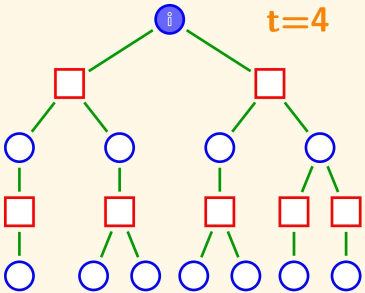


Graph

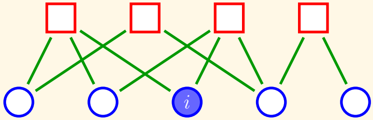


Computation tree

Computation Tree



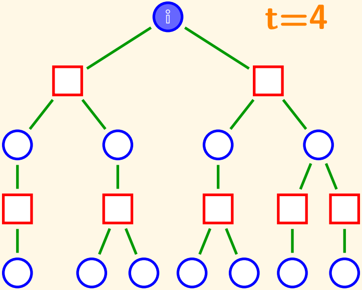
Graph



$$\overbrace{\sum_{\substack{a \in \text{factors} \\ \text{graph}}} f_a(x_{\partial a})}^{f(x)}$$

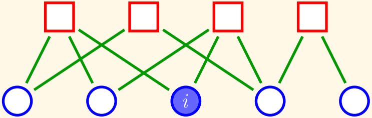
Computation tree

Computation Tree



$$\overbrace{\sum_{\mathbb{a}\in\text{factors tree}} f_{\mathbb{a}}(\mathbb{x}_{\partial\mathbb{a}})}^{\mathbb{f}(\mathbb{x})}$$

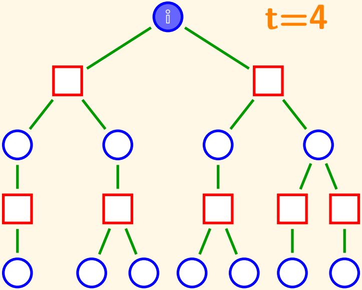
Graph



$$\overbrace{\sum_{a\in\text{factors graph}} f_a(x_{\partial a})}^{f(x)}$$

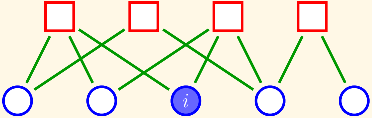
Computation tree

Computation Tree



$$\mathbb{x}^\star := \arg\min_{\mathbb{x}} \overbrace{\sum_{\substack{\mathbb{a}\in\text{factors}\\ \text{tree}}} f_{\mathbb{a}}(\mathbb{x}_{\partial\mathbb{a}})}^{\mathbb{f}(\mathbb{x})}$$

Graph



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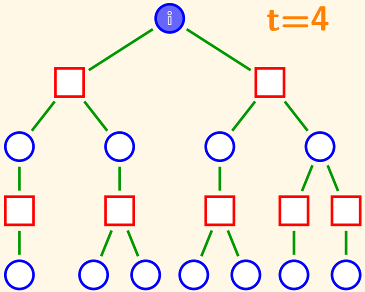
Computation tree

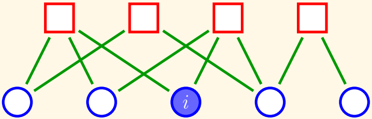
Computation Tree



$$\mathbb{x}^\star := \arg\min_{\mathbb{x}} \overbrace{\sum_{\substack{\mathbb{a}\in\text{factors}\\ \text{tree}}} f_{\mathbb{a}}(\mathbb{x}_{\partial\mathbb{a}})}^{\mathbb{f}(\mathbb{x})}$$

Lemma: $\boxed{\hat{x}_i^{(t)} = \mathbb{x}_{\mathbb{i}}^\star}$

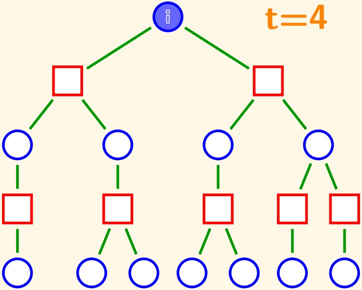
Graph



$$x^\star := \arg\min_{x} \overbrace{\sum_{\substack{a\in\text{factors}\\ \text{graph}}} f_a(x_{\partial a})}^{f(x)}$$

Computation tree

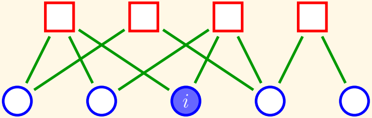
Computation Tree



$$\mathbb{x}^\star := \arg\min_{\mathbb{x}} \overbrace{\sum_{\substack{\mathbb{a}\in\text{factors}\\ \text{tree}}} f_{\mathbb{a}}(\mathbb{x}_{\partial\mathbb{a}})}^{\mathbb{f}(\mathbb{x})}$$

Lemma: $\boxed{\hat{x}_i^{(t)} = \mathbb{x}_{\mathbb{i}}^\star}$

Graph

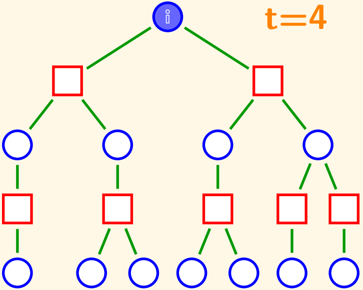


$$x^\star := \arg\min_{x} \overbrace{\sum_{\substack{a\in\text{factors}\\ \text{graph}}} f_a(x_{\partial a})}^{f(x)}$$

$x_i^\star$

Computation tree

Computation Tree

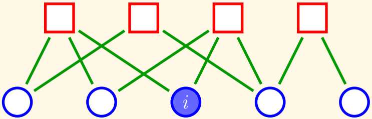


$$\mathbb{x}^\star := \arg\min_{\mathbb{x}} \overbrace{\sum_{\substack{\mathbb{a}\in\text{factors}\\ \text{tree}}} f_{\mathbb{a}}(\mathbb{x}_{\partial\mathbb{a}})}^{\mathbb{f}(\mathbb{x})}$$

?

Lemma: $\boxed{\hat{x}_i^{(t)} = \mathbb{x}_{\mathbb{i}}^\star}$ → $x_i^\star$

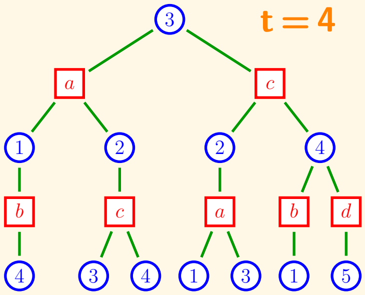
Graph



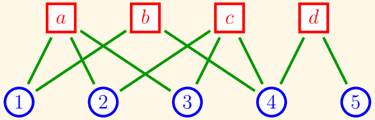
$$x^\star := \arg\min_{x} \overbrace{\sum_{\substack{a\in\text{factors}\\ \text{graph}}} f_a(x_{\partial a})}^{f(x)}$$

Correctness

Computation Tree

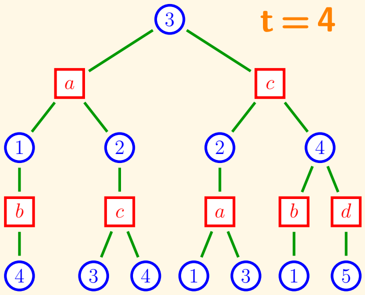


Graph

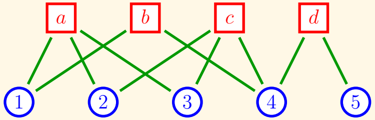


Correctness

Computation Tree



Graph

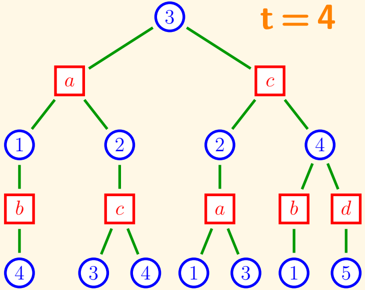


Optimality condition:

$$\overbrace{\sum_{a \in \partial j} \frac{d}{dx_j} f_a(x^\star_{\partial a})}^{\frac{d}{dx_j} f(x)} = 0 \qquad \forall j$$

Correctness

Computation Tree



Optimality condition:

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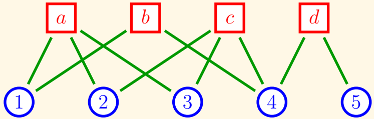
Graph

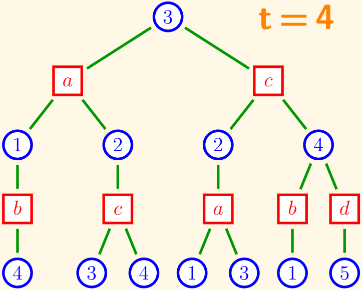


Optimality condition:

$$\overbrace{\sum_{a \in \partial j} \frac{d}{dx_j} f_a(x^\star_{\partial a})}^{\frac{d}{dx_j} f(x)} = 0 \qquad \forall j$$

Correctness

Computation Tree

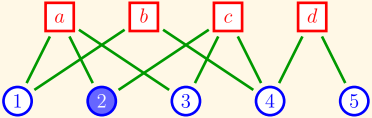


t = 4

Optimality condition:

$$\overbrace{\sum_{a\in\partial j}\frac{d}{dx_j}f_a(x^\star_{\partial a})}^{\frac{d}{dx_j}f(x)} = 0 \qquad \forall j$$

Graph



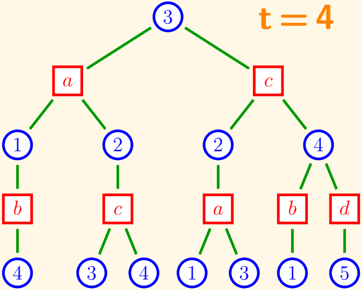
$$\frac{d f_a(x_1^\star, x_2^\star, x_3^\star)}{dx_2} + \frac{d f_c(x_2^\star, x_3^\star, x_4^\star)}{dx_2} = 0 \quad j = 2$$

Optimality condition:

$$\overbrace{\sum_{a\in\partial j}\frac{d}{dx_j}f_a(x^\star_{\partial a})}^{\frac{d}{dx_j}f(x)} = 0 \qquad \forall j$$

Correctness

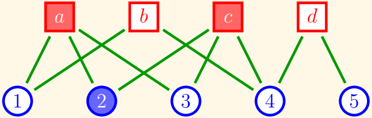
Computation Tree



Optimality condition:

$$\overbrace{\sum_{a \in \partial j} \frac{d}{dx_j} f_a(x^\star_{\partial a})}^{\frac{d}{dx_j} f(x)} = 0 \qquad \forall j$$

Graph



$$\frac{d f_a(x_1^\star, x_2^\star, x_3^\star)}{dx_2} + \frac{d f_c(x_2^\star, x_3^\star, x_4^\star)}{dx_2} = 0 \quad j = 2$$

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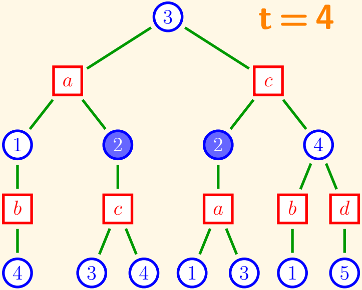
$$\overbrace{\sum_{a \in \partial j} \frac{d}{dx_j} f_a(x^\star_{\partial a})}^{\frac{d}{dx_j} f(x)} = 0 \qquad \forall j$$

Correctness

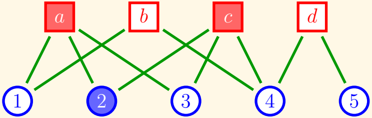
Computation Tree



Optimality condition:

$$\overbrace{\sum_{\mathbb{a} \in \partial \mathbb{j}} \frac{d}{d\mathbb{x}_{\mathbb{j}}} f_{\mathbb{a}}(\mathbb{x}^{\star}_{\partial \mathbb{a}})}^{\frac{d}{d\mathbb{x}_{\mathbb{j}}} \mathbb{f}(\mathbb{x})} = 0 \qquad \forall \mathbb{j}$$

Graph



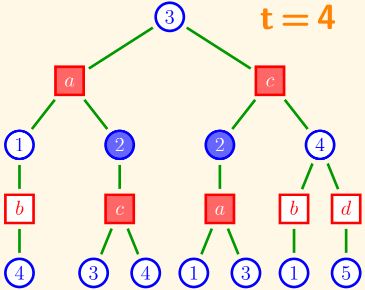
$$\frac{d f_a(x_1^\star, x_2^\star, x_3^\star)}{dx_2} + \frac{d f_c(x_2^\star, x_3^\star, x_4^\star)}{dx_2} = 0 \quad j = 2$$

Optimality condition:

$$\overbrace{\sum_{a \in \partial j} \frac{d}{dx_j} f_a(x^{\star}_{\partial a})}^{\frac{d}{dx_j} f(x)} = 0 \qquad \forall j$$

Correctness

Computation Tree

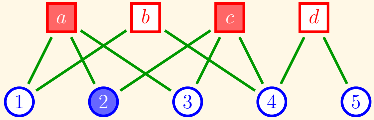


t = 4

Optimality condition:

$$\overbrace{\sum_{a \in \partial j} \frac{d}{dx_j} f_a(x^\star_{\partial a})}^{\frac{d}{dx_j} f(x)} = 0 \qquad \forall j$$

Graph



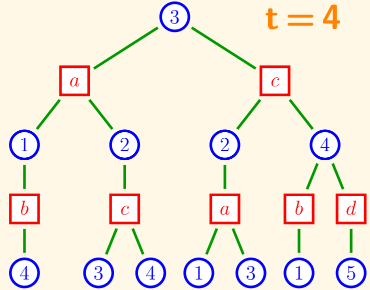
$$\frac{d f_a(x^\star_1, x^\star_2, x^\star_3)}{dx_2} + \frac{d f_c(x^\star_2, x^\star_3, x^\star_4)}{dx_2} = 0 \quad j = 2$$

Optimality condition:

$$\overbrace{\sum_{a \in \partial j} \frac{d}{dx_j} f_a(x^\star_{\partial a})}^{\frac{d}{dx_j} f(x)} = 0 \qquad \forall j$$

Correctness

Computation Tree



Optimality condition:

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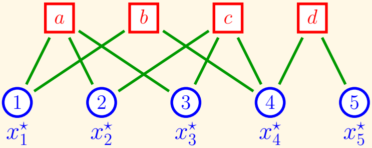
Graph

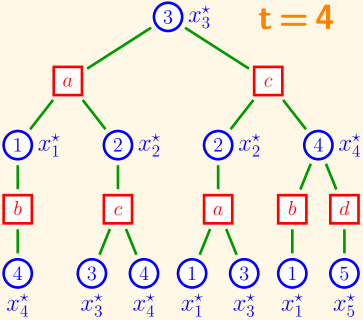


Optimality condition:

$$\overbrace{\sum_{a \in \partial j} \frac{d}{dx_j} f_a(x^\star_{\partial a})}^{\frac{d}{dx_j} f(x)} = 0 \qquad \forall j$$

Correctness

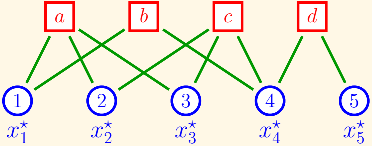
Computation Tree



Optimality condition:

$$\overbrace{\sum_{a\in\partial j} \frac{d}{dx_j} f_a(x^\star_{\partial a})}^{\frac{d}{dx_j} f(x)} = 0 \qquad \forall j$$

Graph

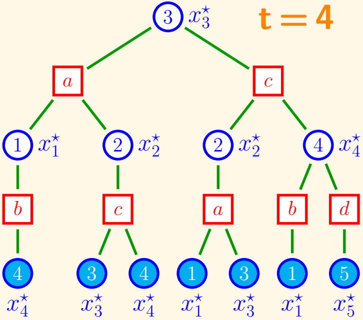


Optimality condition:

$$\overbrace{\sum_{a\in\partial j} \frac{d}{dx_j} f_a(x^\star_{\partial a})}^{\frac{d}{dx_j} f(x)} = 0 \qquad \forall j$$

Correctness

Computation Tree

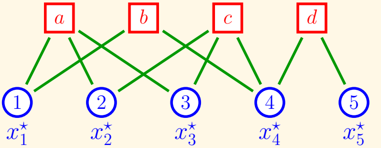


Optimality condition:

$$\overbrace{\sum_{\mathbb{a}\in\partial \mathbb{j}} \frac{d}{d\mathbb{x}_{\mathbb{j}}} f_{\mathbb{a}}(\mathbb{x}^\star_{\partial \mathbb{a}})}^{\frac{d}{d\mathbb{x}_{\mathbb{j}}} \mathbb{f}(\mathbb{x})} = 0 \qquad \forall \mathbb{j}$$

←

Graph

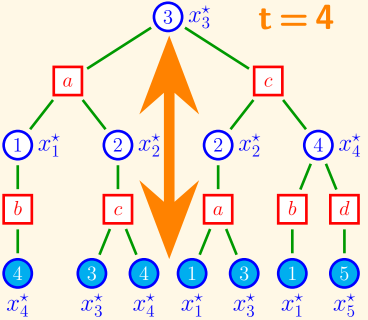


Optimality condition:

$$\overbrace{\sum_{a\in\partial j} \frac{d}{dx_j} f_a(x^\star_{\partial a})}^{\frac{d}{dx_j} f(x)} = 0 \qquad \forall j$$

Correctness

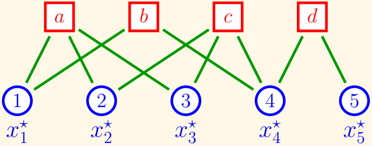
Computation Tree



Optimality condition:

$$\overbrace{\sum_{\mathbb{a} \in \partial \mathbb{j}} \frac{d}{d\mathbb{x}_{\mathbb{j}}} f_{\mathbb{a}}(\mathbb{x}^{\star}_{\partial \mathbb{a}})}^{\frac{d}{d\mathbb{x}_{\mathbb{j}}} \mathbb{f}(\mathbb{x})} = 0 \qquad \forall \mathbb{j}$$

Graph

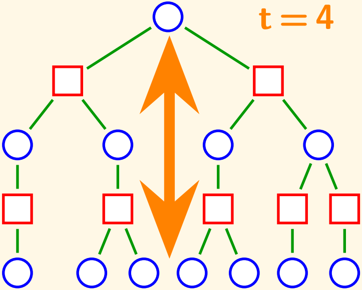


Optimality condition:

$$\overbrace{\sum_{a \in \partial j} \frac{d}{dx_j} f_a(x^{\star}_{\partial a})}^{\frac{d}{dx_j} f(x)} = 0 \qquad \forall j$$

Convergence

Computation Tree

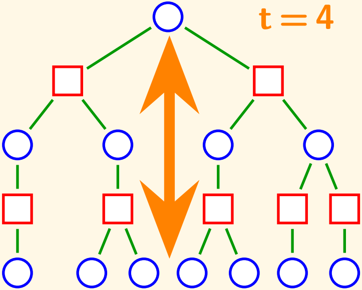


Convergence condition:

$$\sum_{j\in V\setminus i} \omega_j \left|\frac{\partial^2}{\partial x_i \partial x_j} f(x)\right| \le \lambda\, \omega_i \frac{\partial^2}{\partial x_i^2} f(x) \quad \forall i, x$$

Convergence

Computation Tree



Convergence condition:

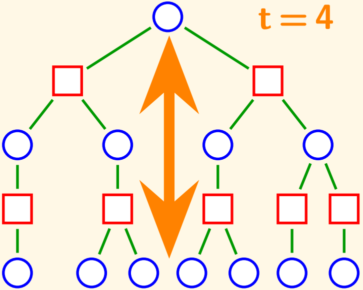
$$\sum_{\mathbb{j}\in\mathbb{V}\setminus\mathbb{i}} \omega_{\mathbb{j}} \left| \frac{\partial^2}{\partial x_{\mathbb{i}} \partial x_{\mathbb{j}}} \mathbb{f}(x) \right| \le \lambda\, \omega_{\mathbb{i}} \frac{\partial^2}{\partial x_{\mathbb{i}}^2} \mathbb{f}(x) \qquad \forall \mathbb{i}, x$$

$$\boxed{\left| \frac{\partial}{\partial p_{j\to a}} x^{\star}(p)_{\mathbb{i}} \right| \le \frac{\max_{\mathbb{i}\in\mathbb{V}} \omega_{\mathbb{i}}}{\min_{\mathbb{i}\in\mathbb{V}} \omega_{\mathbb{i}}} \frac{\lambda^t}{1-\lambda}}$$

(Algorithmic) Locality

Convergence

Computation Tree

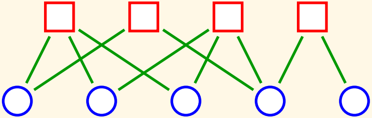


Convergence condition:

$$\sum_{\mathbb{j} \in \mathbb{V} \setminus \mathbb{i}} \omega_{\mathbb{j}} \left| \frac{\partial^2}{\partial \mathbb{x}_{\mathbb{i}} \partial \mathbb{x}_{\mathbb{j}}} \mathbb{f}(\mathbb{x}) \right| \leq \lambda \, \omega_{\mathbb{i}} \frac{\partial^2}{\partial \mathbb{x}_{\mathbb{i}}^2} \mathbb{f}(\mathbb{x}) \quad \forall \mathbb{i}, \mathbb{x}$$

$$\boxed{\left| \frac{\partial}{\partial p_{j \to a}} \mathbb{x}^{\star}(p)_{\mathbb{i}} \right| \leq \frac{\max_{\mathbb{i} \in \mathbb{V}} \omega_{\mathbb{i}}}{\min_{\mathbb{i} \in \mathbb{V}} \omega_{\mathbb{i}}} \, \frac{\lambda^t}{1 - \lambda}}$$

Graph



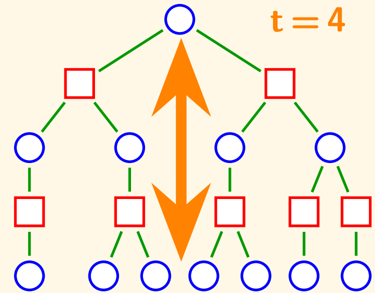
Convergence condition:

$$\sum_{j \in V \setminus i} \omega_j \left| \frac{\partial^2}{\partial x_i \partial x_j} f(x) \right| \leq \lambda \, \omega_i \frac{\partial^2}{\partial x_i^2} f(x) \quad \forall i, x$$

(λ, ω)-scaled diagonal dominance

Convergence

Computation Tree



Convergence condition:

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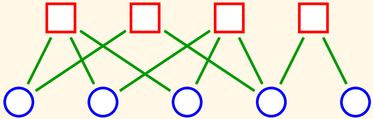
Graph



Convergence condition:

$$\sum_{j\in V\setminus i} \omega_{j} \left|\frac{\partial^2}{\partial x_{i}\partial x_{j}} f(x)\right| \le \lambda\,\omega_{i} \frac{\partial^2}{\partial x_{i}^2}f(x) \quad \forall i, x$$

(λ, ω)-scaled diagonal dominance

Scaled diagonal dominance or walk summability

Computation Tree

$$\sum_{\mathbb{j}\in\mathbb{V}\setminus\mathbb{i}} \omega_{\mathbb{j}} \left|\frac{\partial^2}{\partial \mathbb{x}_{\mathbb{i}}\partial \mathbb{x}_{\mathbb{j}}}\mathbb{f}(\mathbb{x})\right| \le \lambda\,\omega_{\mathbb{i}}\frac{\partial^2}{\partial \mathbb{x}_{\mathbb{i}}^2}\mathbb{f}(\mathbb{x}) \quad \forall \mathbb{i},\mathbb{x}$$

Graph

$$\sum_{j\in V\setminus i} \omega_{j} \left|\frac{\partial^2}{\partial x_{i}\partial x_{j}}f(x)\right| \le \lambda\,\omega_{i}\frac{\partial^2}{\partial x_{i}^2}f(x) \quad \forall i,x$$

Scaled diagonal dominance or walk summability

Computation Tree

$$\sum_{\mathbb{j} \in \mathbb{V} \setminus \mathbb{i}} \omega_{\mathbb{j}} \left| \frac{\partial^2}{\partial \mathbb{x}_{\mathbb{i}} \partial \mathbb{x}_{\mathbb{j}}} \mathbb{f}(\mathbb{x}) \right| \leq \lambda \, \omega_{\mathbb{i}} \frac{\partial^2}{\partial \mathbb{x}_{\mathbb{i}}^2} \mathbb{f}(\mathbb{x}) \quad \forall \mathbb{i}, \mathbb{x}$$

←

Graph

$$\sum_{j \in V \setminus i} \omega_j \left| \frac{\partial^2}{\partial x_i \partial x_j} f(x) \right| \leq \lambda \, \omega_i \frac{\partial^2}{\partial x_i^2} f(x) \quad \forall i, x$$

⇕

$$\rho(|R(x)|) < 1$$

with $R(x) := I - D^{-1/2} \nabla^2 f(x) D^{-1/2}$

Scaled diagonal dominance or walk summability

Computation Tree

$$\sum_{\mathbb{j} \in \mathbb{V} \setminus \mathbb{i}} \omega_{\mathbb{j}} \left| \frac{\partial^2}{\partial \mathbb{x}_{\mathbb{i}} \partial \mathbb{x}_{\mathbb{j}}} \mathbb{f}(\mathbb{x}) \right| \leq \lambda \, \omega_{\mathbb{i}} \frac{\partial^2}{\partial \mathbb{x}_{\mathbb{i}}^2} \mathbb{f}(\mathbb{x}) \quad \forall \mathbb{i}, \mathbb{x}$$

⇕

$$\rho(|\mathbb{R}(\mathbb{x})|) < 1$$

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←

Graph

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Scaled diagonal dominance or walk summability

Computation Tree

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$\leftarrow$

Graph

$$\sum_{j\in V\setminus i} \omega_j \left| \frac{\partial^2}{\partial x_i \partial x_j} f(x) \right| \leq \lambda\, \omega_i \frac{\partial^2}{\partial x_i^2} f(x) \quad \forall i, x$$

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Limitations:

Scaled diagonal dominance or walk summability

Computation Tree

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Graph

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Limitations:

- Inheritance does not capture convergence behavior on the **tree**.

Scaled diagonal dominance or walk summability

Computation Tree

$$\sum_{\mathbb{j}\in\mathbb{V}\setminus\mathbb{i}} \omega_{\mathbb{j}} \left| \frac{\partial^2}{\partial \mathbb{x}_{\mathbb{i}} \partial \mathbb{x}_{\mathbb{j}}} \mathbb{f}(\mathbb{x}) \right| \le \lambda\, \omega_{\mathbb{i}} \frac{\partial^2}{\partial \mathbb{x}_{\mathbb{i}}^2} \mathbb{f}(\mathbb{x}) \quad \forall \mathbb{i}, \mathbb{x}$$

$\Updownarrow$

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Graph

$$\sum_{j\in V\setminus i} \omega_j \left| \frac{\partial^2}{\partial x_i \partial x_j} f(x) \right| \le \lambda\, \omega_i \frac{\partial^2}{\partial x_i^2} f(x) \quad \forall i, x$$

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(Weiss, Freeman, 2001) (Malioutov, Johnson, Willsky, 2006) (Moallemi, Van Roy, 2010)

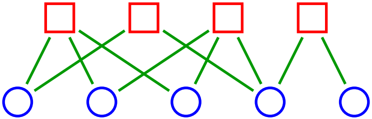
Limitations:

- Inheritance does not capture convergence behavior on the **tree**.
- Condition can **not** be applied to **constrained problems**:

$$\begin{aligned} \text{minimize} \quad & \sum_a f_a(x_{\partial a}) \\ \text{subject to} \quad & Ax = b \end{aligned}$$

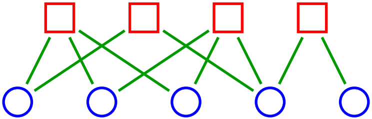
Constrained problems

minimize $\sum_{a} f_a(x_{\partial a})$



Constrained problems

minimize $\sum_{a} f_a(x_{\partial a})$

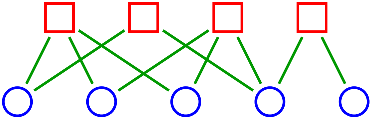


minimize $\sum_{a} f_a(x_{\partial a})$

subject to $h_b(x_{\partial b}) = 0, \quad \forall b$

Constrained problems

minimize $\sum_a f_a(x_{\partial a})$

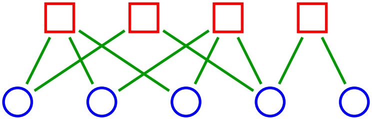


minimize $\sum_a f_a(x_{\partial a}) + \sum_b \chi(h_b(x_{\partial b}))$

$$\chi(z) := \begin{cases} 0 & \text{if } z = 0 \\ \infty & \text{if } z \neq 0 \end{cases}$$

Constrained problems

minimize $\sum_{a} f_a(x_{\partial a})$



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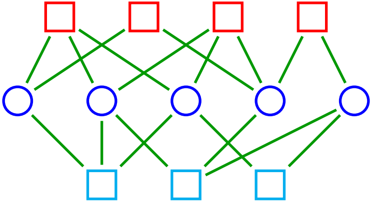
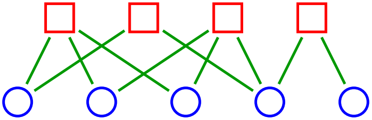


Constrained problems

minimize $\sum_{a} f_a(x_{\partial a})$



minimize $\overbrace{\sum_{a} f_a(x_{\partial a}) + \sum_{b} \chi(h_b(x_{\partial b}))}^{g(x)}$

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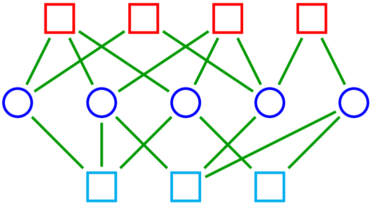
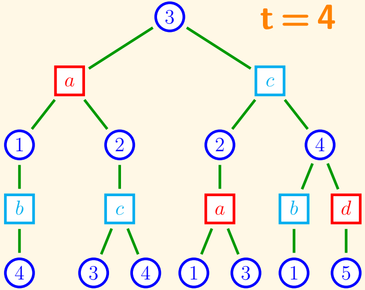


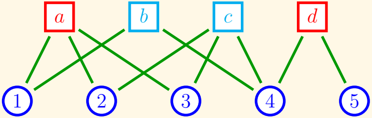
We can run Message Passing. Need different analysis!

Correctness (with constraints!)

Computation Tree



Graph



Correctness (with constraints!)

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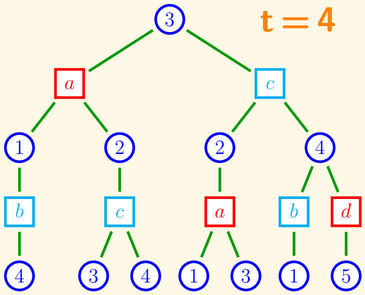
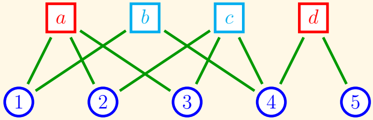


Graph



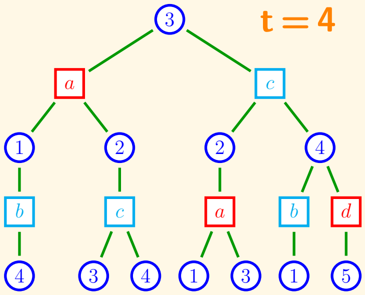
KKT optimality conditions:

$$\sum_{a \in \partial j} \frac{d}{dx_j} f_a(x^\star_{\partial a}) + \sum_{b \in \partial j} \nu^\star_b \frac{d}{dx_j} h_b(x^\star_{\partial b}) = 0 \quad \forall j$$

$$h_b(x^\star_{\partial b}) = 0 \quad \forall b$$

Correctness (with constraints!)

Computation Tree



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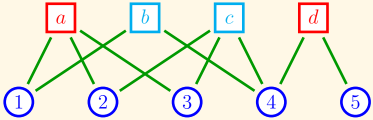

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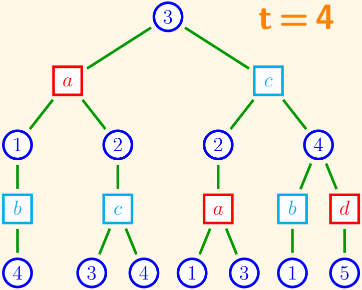
Correctness (with constraints!)

Computation Tree



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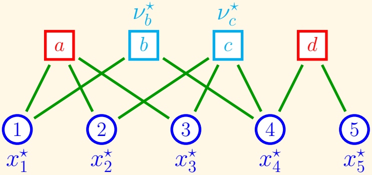
Graph



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Correctness (with constraints!)

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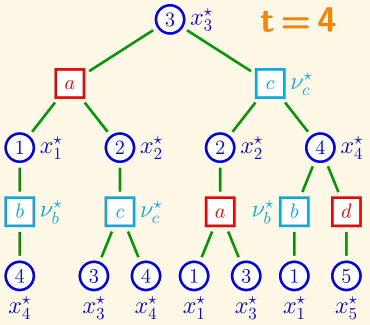

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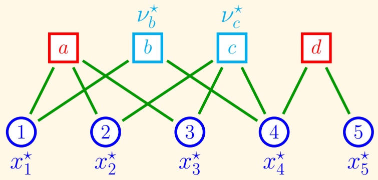

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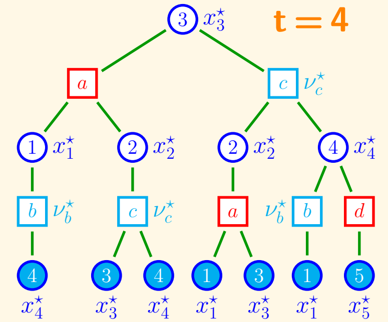

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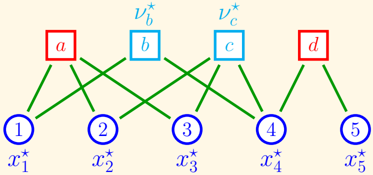

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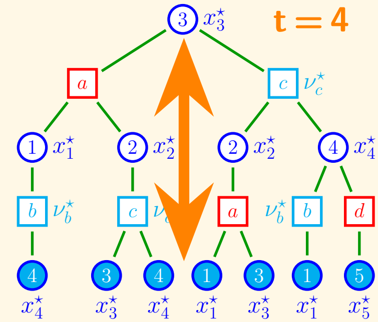
Computation Tree



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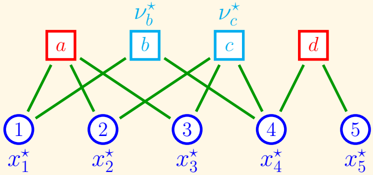

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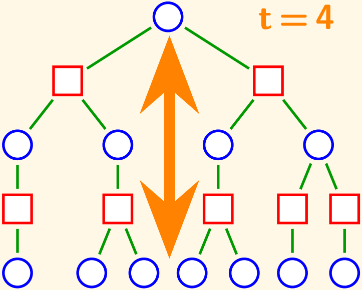
Convergence (with constraints!)

Computation Tree

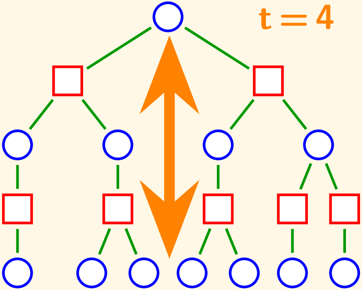


Object of interest:

$$\boxed{\frac{\partial}{\partial p_{j\to a}} \mathbb{x}^{\star}(p)_{\mathbb{i}}}$$

Convergence (with constraints!)

Computation Tree



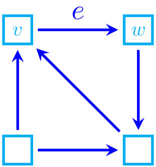
Object of interest:

$$\frac{\partial}{\partial p_{j\to a}} \mathbb{x}^\star(p)_{\mathbb{i}}$$

Beyond scale-diagonal dominance?

Application
Network Flows & Laplacian Solvers

Network Flows



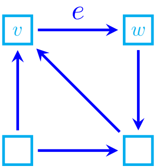
Directed graph $G = (V, E)$

$$\text{minimize} \quad \sum_{e \in E} \overbrace{f_e(x_e)}^{(x_e)^2}$$

$$\text{subject to} \quad Ax = b$$

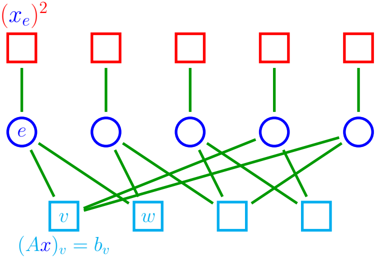
$$A_{ve} := \begin{cases} 1 & \text{if } e \text{ leaves } v, \\ -1 & \text{if } e \text{ enters } v, \\ 0 & \text{otherwise.} \end{cases}$$

Network Flows



Directed graph $G = (V, E)$

$$\begin{array}{ll} \text{minimize} & \sum_{e \in E} \overbrace{f_e(x_e)}^{(x_e)^2} \\ \text{subject to} & Ax = b \end{array}$$



$$A_{ve} := \begin{cases} 1 & \text{if } e \text{ leaves } v, \\ -1 & \text{if } e \text{ enters } v, \\ 0 & \text{otherwise.} \end{cases}$$

Connection to Laplacian solvers

- **Dual problem:** $\min_{\nu \in \mathbb{R}^V} \frac{\nu^T L \nu}{2} - \nu^T b$. **Laplacian:** $\nu^T L \nu = \sum_{\{v,w\} \in E} (\nu_v - \nu_w)^2$
- Solving $L\nu^\star = b$ is key for:
  - PDEs via Finite Element Method
  - Interior Point Methods for Optimization
  - Learning on graphs
  - Faster flow algorithms
  - Graph partitioning
  - Sampling random spanning trees
  - Graph sparsification
- State-of-the-art algorithms (Spielman-Teng, 2004 — Gödel price 2015):
  - **quasi-linear** $O(|E| \log^c(|V|) \log \frac{1}{\varepsilon})$;
  - **centralized**;
  - **involved** (randomized, many graph-theoretic constructions).

Q: What about message-passing?

Message Passing

Algorithm 5: Min-sum, flow problem, quadratic messages, no leaves

Input: Initial messages $\{R^0_{e\to v}\}, \{r^0_{e\to v}\}$, $e \in \vec{E}, v \in \partial e$;

for $s \in \{1, \dots, t\}$ **do**

Compute, for each $e \in \vec{E}$, $v \in \partial e$, with $w = \partial e \setminus v$

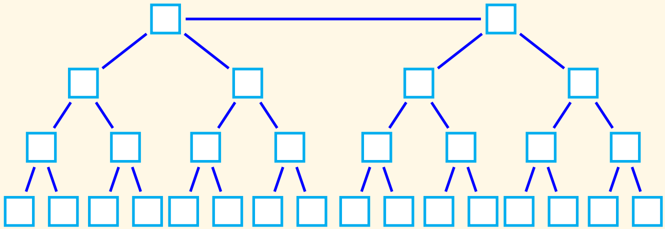
$$R^s_{e\to v} = R_{ee} + \frac{1}{\sum_{f \in \partial w \setminus e} 1/R^{s-1}_{f\to w}}, \qquad r^s_{e\to v} = -A_{we}\frac{\sum_{f \in \partial w \setminus e} A_{wf} r^{s-1}_{f\to w}/R^{s-1}_{f\to w} + b_w}{\sum_{f \in \partial w \setminus e} 1/R^{s-1}_{f\to w}};$$

Output: $\hat{x}^t_e = -\frac{r^t_{e\to v} + r^t_{e\to w}}{R^t_{e\to v} + R^t_{e\to w} - R_{ee}}$, for $e = (v, w) \in \vec{E}$.

Distributed, Simple

Results

Computation Tree for any d-regular graph



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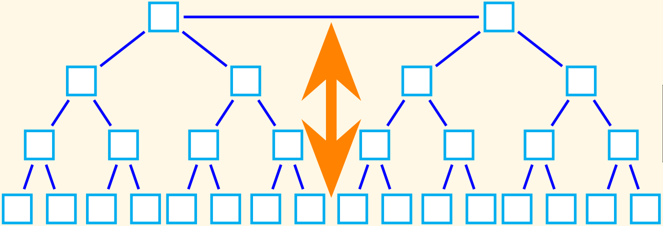
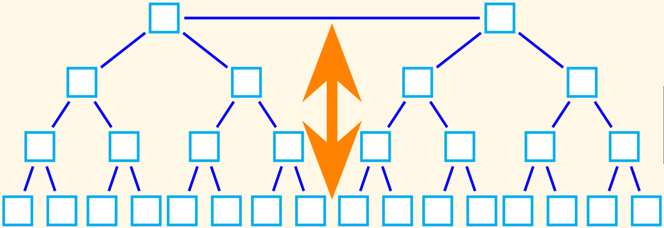


$$\frac{\partial}{\partial p_{e\to v}} \mathbb{x}^{\star}(p)_{\mathbb{e}}$$

Results

Computation Tree for any d-regular graph



$$\frac{\partial}{\partial p_{e\to v}} \mathbb{x}^\star(p)_{\mathbb{e}}$$

Theorem (Rebeschini, Tatikonda 2016)

$$x^\star_{(v,w)} - \hat{x}^{(t)}_{(v,w)} = \sum_{z \in V} \left(P^{(t)}_{vz} - P^{(t)}_{wz} \right) \nu^\star_z$$

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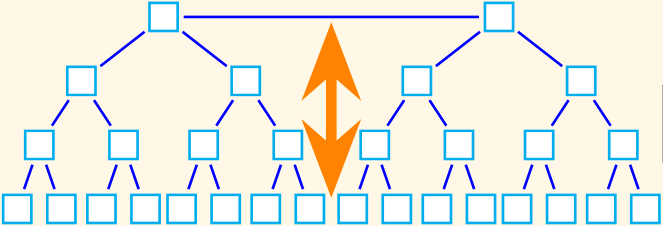

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Corollary

$$\|x^{\star} - \hat{x}^{(t)}\|_{\infty} \le \|\nu^{\star}\|_{\infty} \sum_{z\in V} \left|P^{(t)}_{vz} - P^{(t)}_{wz}\right|$$

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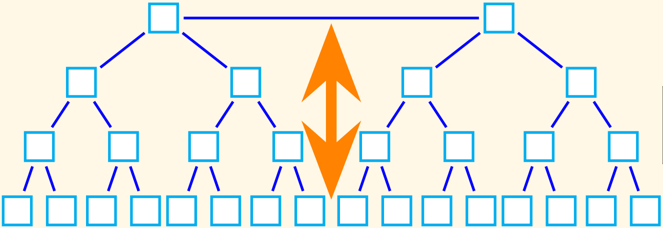

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$$\|x^\star - \hat{x}^{(t)}\|_\infty \le \|\nu^\star\|_\infty \sum_{z\in V} \left|P^{(t)}_{vz} - P^{(t)}_{wz}\right|$$

For some graphs (rings, grids, etc)

$$\max_{\{v,w\}\in E} \sum_{z\in V} \left|P^{(t)}_{vz} - P^{(t)}_{wz}\right| \sim \frac{1}{t^\gamma}$$

Hence, $O\left(|E| \frac{1}{\varepsilon^{1/\gamma}}\right)$

MAIN MESSAGE:

General toolbox for message passing in convex optimization.

- **General framework** for message-passing with constraints.
- **Simple, distributed, fast** algorithm for Laplacian and Network Flows algorithms.

A new approach to Laplacian solvers and flow problems, **arXiv:1611.07138**