The strong slope conjecture for twisted generalized Whitehead doubles

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Modular Forms and Quantum Knot Invariants
BIRS
12 March, 2018

Jones slope

$J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1}]$: the colored Jones polynomial for a knot K in S^3

[Garoufalidis],[Garoufalidis-Le]

$\delta_K(n)$ ($\delta_K^*(n)$) : the maximum (minimum) degree of $J_{K,n}(q)$

For large n,

$$\delta_K(n) = a(n)n^2 + b(n)n + c(n) \quad (a(n), b(n), c(n) \in \mathbb{Q}, \text{ periodic})$$

$$\delta_K^*(n) = a^*(n)n^2 + b^*(n)n + c^*(n) \quad (a^*(n), b^*(n), c^*(n) \in \mathbb{Q}, \text{ periodic})$$

$$js(K) := \left\{4a(n) \mid n \in \mathbb{N}\right\}, \quad js^*(K) := \left\{4a^*(n) \mid n \in \mathbb{N}\right\} \quad (\text{Jones slope})$$

$$jx(K) := \left\{2b(n) \mid n \in \mathbb{N}\right\}, \quad jx^*(K) := \left\{2b^*(n) \mid n \in \mathbb{N}\right\}.$$

Boundary slope

K : a knot in S^3 $\quad E(K)$: the exterior $S^3 - \mathrm{int}N(K)$
(μ, λ) : the preferred meridian-longitude pair of K

Any homotopically nontrivial simple closed curves in $\partial E(K)$ represents $p[\mu] + q[\lambda] \in H_1(\partial E(K))$ for some relatively prime integers p and q.

$p/q \in \mathbb{Q} \cup \{\infty\}$ is a boundary slope of K
$\Leftrightarrow \exists$ an *essential* (i.e. orientable, incompressible and boundary-incompressible) surface F in $E(K)$ s.t. a component of ∂F represents $p[\mu] + q[\lambda] \in H_1(\partial E(K))$.

Ex. A minimal genus Seifert surface F of K is an essential surface and the boundary slope of F is 0.

$$bs(K) := \Big\{ r \in \mathbb{Q} \cup \{\infty\} \mid r \text{ is a boundary slope of } K \Big\}$$

Remark. $0 \in bs(K)$

Conjecture (**Slope conjecture (Garouflidis)**)

$js(K) \cup js^*(K) \subset bs(K)$.

Conjecture (**Strong slope conjecture (Kalfagianni-Tran)**)

Given $p/q \in js(K)$, with $q > 0$ and $(p,q) = 1$, there exists an essential surface $S \subset E(K)$ with $|\partial S|$ boundary components such that each component of ∂S has slope p/q, and

$$\frac{\chi(S)}{|\partial S| q} \in jx(K).$$

Similarly, given $p^*/q^* \in js^*(K)$, with $q^* > 0$ and $(p^*, q^*) = 1$, there exists an essential surface $S^* \subset E(K)$ with $|\partial S^*|$ boundary components such that each component of ∂S^* has slope p^*/q^*, and

$$-\frac{\chi(S^*)}{|\partial S^*| q^*} \in jx^*(K).$$

Following Lee and van der Veen, we present a sharpened version that more directly yokes the two conjecture together.

Conjecture

$\delta_K(n) = a(n)n^2 + b(n)n + c(n)$ *for a large* n. *Then for each* n *there is an essential suface* F_n *in the exterior of* K *such that*

- *(Slope Conjecture)* $4a(n)$ *is the boundary slope of* F_n.
- *(Strong Slope Conjecture)* $4a(n) = p/q$ *for coprime integers* p, q *with* $q > 0$,

$$2b(n) = \frac{\chi(F_n)}{|\partial F_n| q}$$

A similar statement holds for the minimum degree.

Known results for Slope conjecture

- torus knots, alternating knots, non-alternating knots with up to 9 crossings, $(-2, 3, p)$ pretzel knots (Garoufalidis)
- adequate knots (Futer-Kalfagianni-Purcell)
- 2-parameter family of 2-fusion knots (Garoufalidis-van der Veen)
- iterated cables of adequate knots, cables of 2–fusion knots (Kalfagianni-Tran)
- graph knots (Motegi-T)
- certain families of 3-tangle pretzel knots (C.R.S. Lee-van der Veen)
- certain families of Montesinos knots (Leng-Yang-Liu)

Known results for Strong Slope conjecture

- iterated cables of adequate knots, $(-2, 3, p)$ pretzel knots (Kalfagianni-Tran)
- 8, 9-crossing non-alternating knots (Kalfagianni-Tran), $(9_{47}, 9_{48}$ Howie)
- certain families of 3-tangle pretzel knots (Lee-van der Veen)
- certain families of Montesinos knots (Leng-Yang-Liu)

Twisted generalized Whitehead double of a knot

Let V be a standardly embedded solid torus in S^3 with a preferred meridian-longitude (μ_V, λ_V), and take a pattern (V, k_ω^τ) where k_ω^τ is a knot in the interior of V. Given a knot K in S^3 with a preferred meridian-longitude (μ_K, λ_K), consider an orientation preserving embedding $f : V \to S^3$ which sends the core of V to a knot $K \subset S^3$ and $f(\mu_V) = \mu_K$ and $f(\lambda_V) = \lambda_K$. Then the image $f(k_\omega^\tau)$ is called a *τ–twisted, ω–generalized Whitehead double of* K, and denoted by $W_\omega^\tau(K)$. W_1^0 is the (untwisted) negative Whitehead double of K.

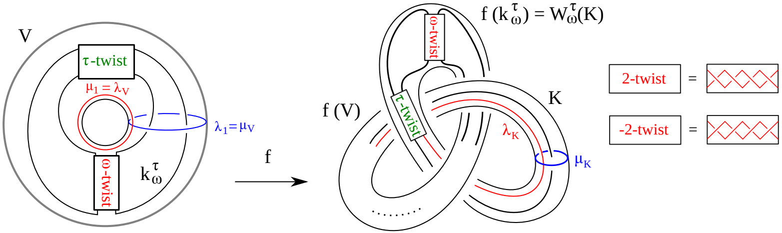


Figure : Twisted generalized Whitehead double of K; $f : V \to S^3$ is a faithful embedding and it maps the core of V to K.

Main theorem

Theorem

Let K be a knot such that $d_+[J_{K,n}(q)]$ and $d_-[J_{K,n}(q)]$ are, for all integers $n \geq 0$, quadratic quasi-polynomials $\delta_K(n) = a(n)n^2 + b(n)n + c(n)$ and $\delta_K^*(n) = a^*(n)n^2 + b^*(n)n + c^*(n)$ of period ≤ 2 with $b(1) \leq 0$ and $b^*(1) \geq 0$.

1. *If K satisfies the slope conjecture, then all of its twisted generalized Whitehead doubles also satisfy the slope conjecture.*
2. *If K satisfies the strong slope conjecture, then all of its twisted generalized Whitehead doubles also satisfy the strong slope conjecture.*

Examples of knots satisfying the hypotheses

1. Torus knots
2. Adequate knots
3. Non-alternating knots with up to 9 crossings except for 8_{20}, 9_{43}, 9_{44}.

Maximum degree of $J_{W^\tau_\omega(K),n}(q)$

Proposition (**maximum-degree**)

Let K be a knot with $\delta_K(n) = a(n)n^2 + b(n)n + c(n)$. We put $a_1 := a(1)$, $b_1 := b(1)$, and $c_1 := c(1)$. We assume that the period of $\delta_K(n)$ is less than or equal to 2 and that $b_1 \leq 0$. Then the maximum degree of the colored Jones polynomial of its twisted generalized Whitehead double $\delta_{W^\tau_\omega(K)}(n) = a_W(n)n^2 + b_W(n)n + c_W(n)$ is given by

$$\delta_{W^\tau_\omega(K)}(n) = \begin{cases} (4a_1 - \tau)n^2 + (-4a_1 + 2b_1 + \tau - \frac{1}{2})n + a_1 - b_1 + c_1 + \frac{1}{2} & (a_1 > \frac{\tau}{4}), \\ -\frac{1}{2}n + a_1 + b_1 + c_1 + \frac{1}{2} & (a_1 \leq \frac{\tau}{4}). \end{cases}$$

Example $K = T_{p,q}$

$$\delta_K(n) = \frac{pq}{4}n^2 - \frac{pq}{4} - (1 + (-1)^n)\frac{(p-2)(q-2)}{8} \text{ (Garoufalidis)}$$

$$\delta_{W^\tau_\omega(K)}(n) = \begin{cases} (pq - \tau)n^2 + (-pq + \tau - \frac{1}{2})n + \frac{1}{2} & (\tau < pq), \\ -\frac{1}{2}n + \frac{1}{2} & (\tau \geq pq). \end{cases}$$

Minimum degree of $J_{W^{\tau}_{\omega}(K),n}(q)$

Proposition (**minimum-degree**)

Let $\delta^*_K(n) = a^*(n)n^2 + b^*(n)n + c^*(n)$. *We put* $a^*_1 := a^*(1)$, $b^*_1 := b^*(1)$, *and* $c^*_1 := c^*(1)$. *We assume that the period of* $\delta^*_K(n)$ *is less than or equal to* 2, $b^*_1 \geq 0$. *Then the minimum degree of the colored Jones polynomial of its twisted generalized Whitehead double* $\delta^*_{W^{\tau}_{\omega}(K)}(n) = a^*_W(n)n^2 + b^*_W(n)n + c^*_W(n)$ *is given by*

$$\delta^*_{W^{\tau}_{\omega}}(n) = \begin{cases} (4a^*_1 - \frac{2\omega-1}{2} - \tau)n^2 + (-4a^*_1 + 2b^*_1 + \omega - 1 + \tau)n + a^*_1 - b^*_1 + c^*_1 + \frac{1}{2} & (a^*_1 < \frac{\tau}{4} - \frac{1}{8}), \\ -\omega n^2 + \frac{2\omega-1}{2}n + a^*_1 + b^*_1 + c^*_1 + \frac{1}{2} & (a^*_1 \geq \frac{\tau}{4} - \frac{1}{8}). \end{cases}$$

Example $K = T_{p,q}$

$\delta^*_K(n) = \dfrac{pq-p-q}{2}n - \dfrac{pq-p-q}{2}$ (Garoufalidis)

$$\delta^*_{W^{\tau}_{\omega}}(n) = \begin{cases} (-\omega + \frac{1}{2} - \tau)n^2 + (pq - p - q + \omega - 1 + \tau)n - pq + p + q + \frac{1}{2} & (\tau > \frac{1}{2}), \\ -\omega n^2 + (\omega - \frac{1}{2})n + \frac{1}{2} & (\tau \leq \frac{1}{2}). \end{cases}$$

Colored Jones polynomial of $W^\tau_\omega(K)$

$J_{W^\tau_\omega(K),n}(q)$: the colored Jones polynomial of a knot $W^\tau_\omega(K)$ for $n \in \mathbb{N}$

$$J'_{W^\tau_\omega(K),n}(q) := \frac{J_{W^\tau_\omega(K),n+1}(q)}{J_{\bigcirc,n+1}(q)}, J_{\bigcirc,n}(q) = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$$

$$< d >= (-1)^d [d+1], \quad [d] = \frac{x^d - x^{-d}}{x - x^{-1}}$$

s, t, u : non-negative integers

s.t. $s + t + u \equiv 0 \pmod 2$, $|s - t| \le u \le s + t$ (*admissible*)

$\delta(u; s, t) = (-1)^{\frac{s+t+u}{2}} x^{\frac{1}{4}(u^2 - s^2 - t^2 + 2u - 2s - 2t)}$

$< s, t, u >= (-1)^{\frac{s+t+u}{2}} \frac{[\frac{s+t+u}{2} + 1]![\frac{t+u-s}{2}]![\frac{u+s-t}{2}]![\frac{s+t-u}{2}]!}{[s]![t]![u]!}$

$$\left\langle \begin{array}{ccc} A & B & E \\ D & C & F \end{array} \right\rangle = \frac{\prod_{i=1}^3 \prod_{j=1}^4 [b_i - a_j]|}{[A]![B]![C]![D]![E]![F]!} \sum_{\max\{a_j\} \le s \le \min\{b_i\}} \frac{(-1)^s [s+1]!}{\prod_{i=1}^3 [b_i - s]| \prod_{j=1}^4 [s - a_j]}$$

$a_1 = \frac{A+B+E}{2}$, $a_2 = \frac{B+D+F}{2}$, $a_3 = \frac{C+D+E}{2}$, $a_4 = \frac{A+C+F}{2}$,
$b_1 = \frac{\Sigma-A-D}{2}$, $b_2 = \frac{\Sigma-E-F}{2}$, $b_3 = \frac{\Sigma-B-C}{2}$

Colored Jones polynomial of $W_\omega^\tau(K)$

Proposition

$$J'_{W_\omega^\tau(K),n}(q) = \frac{1}{<n>} \sum_{j,k=0}^{n} \frac{<2k>}{<n,n,2k>} \frac{<2j>}{<n,n,2j>} \left\langle \begin{matrix} n & n & 2j \\ n & n & 2k \end{matrix} \right\rangle q^{-\omega j(j+1) - \tau k(k+1)} J'_{K,2k}(q)$$

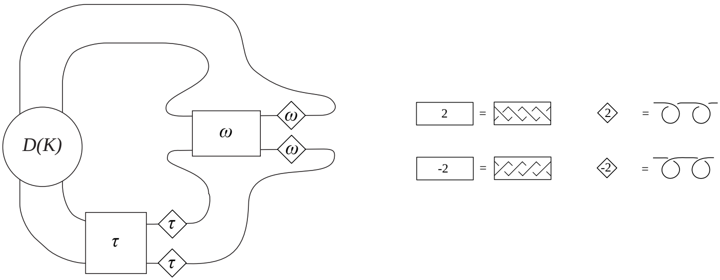
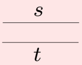
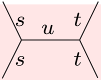
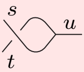


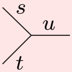
Figure : A diagram of $W_\omega^\tau(K)$ with trivial writhe

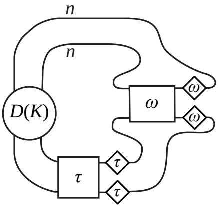
Colored Jones polynomial $W^{\tau}_{\omega}(K)$

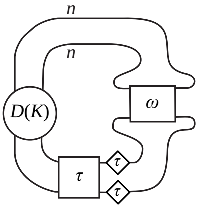


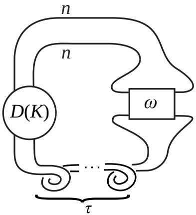
$= \sum_{u} \frac{<u>}{<s,t,u>}$


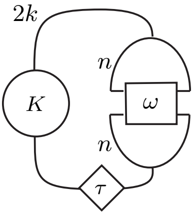


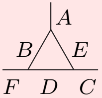
$= \delta(u; s, t)$


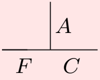
$< n > J'_{W^{\tau}_{\omega}(K),n}(q) =$


$= q^{-\frac{1}{2}\omega n(n+2)}$


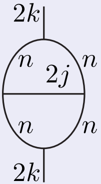
$= q^{-\frac{1}{2}\omega n(n+2)}$


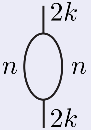
$= q^{-\frac{1}{2}\omega n(n+2)} \sum_{k=0}^{n} \frac{<2k>}{<n,n,2k>}$


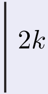


$$= \frac{\left\langle \begin{matrix} A & B & E \\ D & C & F \end{matrix} \right\rangle}{<A,F,C>}$$


Lemma

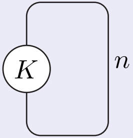


$$= \frac{\left\langle \begin{matrix} n & n & 2j \\ n & n & 2k \end{matrix} \right\rangle}{< n, 2k, n >}$$


$$= \frac{\left\langle \begin{matrix} n & n & 2j \\ n & n & 2k \end{matrix} \right\rangle}{< 2k >}$$


Lemma

For a 0 *framed diagram of any knot* K*:*

$$< n > J'_{K,n}(q) =$$


Proposition (Garoufalidis's convention)

Let $\delta'_K(n) = \alpha(n)n^2 + \beta(n)n + \gamma(n)$*. We put* $\alpha_0 := \alpha(0)$*,* $\beta_0 := \beta(0)$*, and* $\gamma_0 := \gamma(0)$*. We assume that the period of* $\delta'_K(n)$ *is less than or equal to* 2 *and that* $-2\alpha_0 + \beta_0 + \frac{1}{2} \leq 0$*. Then, for suitably large* n*, the maximum degree of the colored Jones polynomial of its twisted generalized Whitehead double is given by*

$$\delta'_{W^\tau_\omega(K)}(n) = \begin{cases} (4\alpha_0 - \tau)n^2 + (2\beta_0 - \tau)n + \gamma_0 & (\alpha_0 > \frac{\tau}{4}), \\ -n + \gamma_0 & (\alpha_0 \leq \frac{\tau}{4}). \end{cases}$$

Outline of proof the case $\tau = 0$, $\omega = 1$ (the negative Whitehead double of K)

$$< n > J'_{W^\tau_\omega(K),n}(q) = \sum_{j,k=0}^{n} f(j,k;q).$$

We prove that in each case of $\alpha_0 > 0$ and $\alpha_0 \leq 0$, there exists a unique pair (j_0, k_0) such that

$$\max_{0 \leq j,k \leq n} d_+[f(j,k;q)] = d_+[f(j_0,k_0;q)]$$

Case $j+k \leq n$

$$d_+[f(j,k;q)] = -j^2 + 4\alpha_0 k^2 + (2\beta_0+1)k - \frac{n}{2} + \gamma_0$$

$$\Rightarrow \max_{0\leq j\leq n-k} d_+[f(j,k;q)] = d_+[f(0,k;q)] = 4\alpha_0 k^2 + (2\beta_0+1)k - \frac{n}{2} + \gamma_0.$$

$$d_+[f(0,k;q)] = 4\alpha_0\left(k + \frac{2\beta_0+1}{8\alpha_0}\right)^2 - \frac{(2\beta_0+1)^2}{16\alpha_0} - \frac{n}{2} + \gamma_0. \quad (\alpha_0 \neq 0)$$

If $\alpha_0 > 0$, since $-\frac{2\beta_0+1}{8\alpha_0} < \frac{n}{2}$ for sufficiently large n, then this is maximized at $k=n$,

$$\max_{0\leq k\leq n} d_+[f(0,k;q)] = d_+[f(0,n;q)] = 4\alpha_0 n^2 + (2\beta_0 + \frac{1}{2})n + \gamma_0.$$

If $\alpha_0 \leq 0$, since $-2\alpha_0 + \beta_0 + \frac{1}{2} \leq 0$ by the assumption, then we have $\beta_0 + \frac{1}{2} \leq 0$. Therefore, if $\alpha_0 < 0$, then $\frac{2\beta_0+1}{8\alpha_0} \geq 0$, and so, this is maximized at $k=0$.

$$\max_{0\leq k\leq n} d_+[f(0,k;q)] = d_+[f(0,0;q)] = -\frac{n}{2} + \gamma_0.$$

If $\alpha_0 = 0$, $\max_{0\leq j\leq n-k} d_+[f(j,k;q)] = d_+[f(0,k;q)] = -\frac{n}{2} + \gamma_0$.

Exteriors of twisted, generalized Whitehead doubles and those of two-bridge links

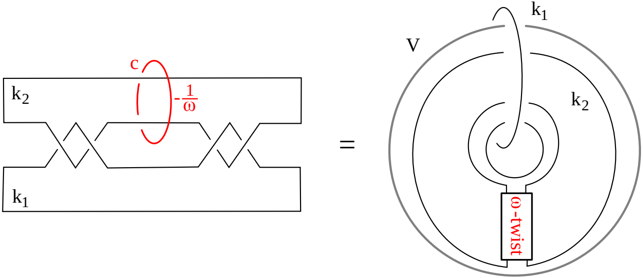


Figure : $k_1 \cup k_2$ is a two bridge link $[2, 2\omega, -2] = \mathcal{L}_{\frac{4\omega-1}{8\omega}}$.

The exterior of $W^{\tau}_{\omega}(K)$ is the union of the exterior $E(K)$ and $V - \mathrm{int}N(k_2)$; the latter is the exterior of the two-bridge link $k_1 \cup k_2$, which is expressed as $[2, 2\omega, -2]$. $[2, 2, -2]$ is the Whitehead double.

(Hatcher-Thurston) correspondence between a certain collection of "minimal edge paths" in the Farey diagram from $1/0$ to p/q and the properly embedded incompressible and ∂–incompressible surfaces with boundary in the exterior of the two bridge knot $\mathcal{L}_{p/q}$

(Floyd-Hatcher) extension of [H-T] to two-bridge links of two components

(Hoste-Shanahan) boundary slopes of such surfaces in [F-H]

$$\Downarrow$$

list of all the properly embedded essential surfaces in the exterior of $\mathcal{L}_{(4\omega-1)/8\omega}$, their Euler characteristics, their boundary slopes, and number of boundary components

Data for essential surfaces in the exterior of the whitehead link $\mathcal{L}_{3/8}$

HS path	branch pattern	χ	boundary slopes $\beta>0$	number of boundary components $\beta>0$
γ_1	$ADAADA$	$-\alpha-\beta$	$(2\frac{\beta}{\alpha}, 2\frac{\alpha}{\beta})$	$(\gcd(2\beta,\alpha), \gcd(2\alpha,\beta))$
γ_2	$ADAADA$	$-\alpha-\beta$	$(0,0)$	(α,β)
γ_3	$ADAADA$	$-\alpha-\beta$	$(0,0)$	(α,β)
γ_5	$ADCDA$	$-\alpha$	$(-2\frac{\beta}{\alpha}, -2\frac{\alpha}{\beta}-2)$	$(\gcd(2\beta,\alpha), \gcd(2\alpha,\beta))$
γ_6	$ABBCBBA$	$-\alpha$	$(-4,-2)$	(α,β)
γ_5'	ACA	$-\alpha$	$(-4+\frac{X}{\beta}, -2-\frac{X}{\beta})$	$(\gcd(X,\alpha), \gcd(X,\beta))$

Theorem (**slope conjecture**)

Let K be a knot such that $\delta_K(n) = a(n)n^2 + b(n)n + d(n)$ and $\delta^*_K(n) = a^*(n)n^2 + b^*(n)n + d^*(n)$ are quadratic quasi-polynomial of period ≤ 2 with $b(1) \leq 0$ and $b^*(1) \geq 0$. If K satisfies the slope conjecture, then its twisted generalized Whitehead double also satisfies the slope conjecture.

Outline of proof

$$\delta_{W_1^0(K)}(n) = \begin{cases} 4a_1 n^2 + (-4a_1 + 2b_1 - \frac{1}{2})n + a_1 - b_1 + c_1 + \frac{1}{2} & (a_1 > 0), \\ -\frac{1}{2}n + a_1 + b_1 + c_1 + \frac{1}{2} & (a_1 \leq 0). \end{cases}$$

$$\delta^*_{W_1^0}(n) = \begin{cases} (4a_1^* - \frac{1}{2})n^2 + (-4a_1^* + 2b_1^*)n + a_1^* - b_1^* + c_1^* + \frac{1}{2} & (a_1^* < -\frac{1}{8}), \\ -n^2 + \frac{1}{2}n + a_1^* + b_1^* + c_1^* + \frac{1}{2} & (a_1^* \geq -\frac{1}{8}). \end{cases}$$

$\Longrightarrow$

$$js_{W_1^0(K)} \subset \{4js(K), 0\} = \{16a_1, 0\},$$
$$js^*_{W_1^0(K)} \subset \{4js^*(K) - 2, -4\} = \{16{a_1}^* - 2, -4\}$$

Let us find essential surfaces in $E(W_1^0(K))$ whose boundary slopes are these Jones slopes!

Realization of the Jones slopes arising from the maximum degree

Case $a_1 > 0$. Since K satisfies the slope conjecture, the Jones slope $4a_1$ is realized by a boundary slope of an essential surface $S_K \subset E(K)$.

Claim

$\exists$ *an essential orientable surface* F_{γ_1} *in* $V - \mathrm{int}N(k_2)$
s.t. each component of $F_{\gamma_1} \cap \partial V$ *has slope* $4a_1$*,*
each component of $F_{\gamma_1} \cap \partial N(k_2)$ *has* $16a_1$

Proof Let us take an essential surface F_{γ_1} in $S^3 - \mathrm{int}N(k_1 \cup k_2) = V - \mathrm{int}N(k_2)$. Then it has a pair of boundary slopes $(2\frac{\beta}{\alpha}, 2\frac{\alpha}{\beta})$ on k_1, k_2. Then F_{γ_1} has boundary slopes $\frac{2\beta}{\alpha}$ on $\partial N(k_1)$ and $\frac{2\alpha}{\beta}$ on $\partial N(k_2)$. Using the preferred meridian-longitude (μ_V, λ_V) instead of (μ_1, λ_1), $F_{\gamma_1} \cap \partial V$ has slope $\frac{\alpha}{2\beta}$. Choose α, β so that $\frac{\alpha}{2\beta} = 4a_1 > 0$, i.e. $\frac{\alpha}{\beta} = 8a_1 > 0$. Then $F_{\gamma_1} \subset V - \mathrm{int}N(k_2)$ has boundary slope $16a_1$ on $\partial N(k_2)$ and $4a_1$ on ∂V.

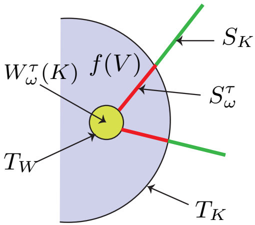
Realization of the Jones slopes arising from the maximum degree

Proof of Theorem for the case $a_1 > 0$

$\exists$ an essential surface S in $E(W_1^0(K))$ whose boundary slope is $16a_1$.

Let us take the image $f(F_{\gamma_1})$ in $X = f(V - \mathrm{int}N(k_2))$, and denote it by S_1^0. Write $T_K = \partial E(K) = f(\partial V)$ and $T_W = \partial N(W_1^0(K)) = f(\partial N(k_2))$. By construction S_1^0 is essential in $f(V - \mathrm{int}N(k_2))$ and each component of $S_1^0 \cap T_K$ has slope $4a_1$ and each component of $S_1^0 \cap T_W$ has slope $16a_1$.

To build a required essential surface $S \subset E(W_1^0(K))$ we take m parallel copies mS_1^0 of the essential surface S_1^0 and n parallel copies nS_K of the essential surface S_K, and then glue them along their boundaries to obtain a connected surface $S = mS_1^0 \cup nS_K$ in $E(W_1^0(K))$.



Strong slope conjecture for twisted generalized Whitehead doubles

Theorem (strong slope conjecture)

Let K be a knot such that $\delta_K(n) = a(n)n^2 + b(n)n + d(n)$ and $\delta_K^*(n) = a^*(n)n^2 + b^*(n)n + d^*(n)$ are quadratic quasi-polynomial of period ≤ 2 with $b(1) \leq 0$ and $b^*(1) \geq 0$. Suppose that K satisfies the strong slope conjecture. Then its twisted generalized Whitehead double of K also satisfies the strong slope conjecture.

Outline of proof

Write $\delta_{W^\tau_\omega(K)}(n) = a_W(n)n^2 + b_W(n)n + c_W(n)$ and $\delta^*_{W^\tau_\omega(K)}(n) = a^*_W(n)n^2 + b^*_W(n)n + c^*_W(n)$. It follows from Propositions that coefficients of $\delta_{W^\tau_\omega(K)}(n)$ and $\delta^*_{W^\tau_\omega(K)}(n)$ are constants and so we may write $a_W(n) = a_W$, $b_W(n) = b_W$, $c_W(n) = c_W$, $a^*_W(n) = a^*_W$, $b^*_W(n) = b^*_W$, $c^*_W(n) = c^*_W$. Then we show that essential surfaces S (and S^*) in $E(W^\tau_\omega(K))$ given in the proof of Theorem(slope conjecture) satisfy the condition of the strong slope conjecture:

$$S \text{ has boundary slope } p/q = 4a_W, \quad \frac{\chi(S)}{|\partial S| q} = 2b_W$$

$$S^* \text{ has boundary slope } p^*/q^* = 4a^*_W, \quad -\frac{\chi(S)}{|\partial S| q^*} = 2b^*_W$$

Jones surfaces arising from the maximum degree.

Case $a_1 > 0$ Write $a_1 = r/s$ where r and s are coprime integers and $s > 0$. Then, as a ratio of coprime integers, the denominator of $4a_1$ is $s/\gcd(4,s)$. Since K satisfies the Strong Slope Conjecture, there is a properly embedded essential surface S_K in the exterior of K whose boundary slope is $4a_1$ and

$$\frac{\chi(S_K)}{|\partial S_K| \cdot \frac{s}{\gcd(4,s)}} = 2b_1.$$

When addressing the Slope Conjecture for $W_1^0(K)$ in this case, we constructed a properly embedded essential surface $S = mS_K \cup nS_1^0$ in the exterior of $W_1^0(K)$ by joining m copies of S_K in $E(K)$ to n copies of the surface S_1^0 in $V - \mathrm{int}N(k_2)$. This requires that

$$m|\partial S_K| = n|\partial S_1^0 \cap T_K|.$$

The surface S_1^0 is identified with a surface of type F_{γ_1} in the exterior of the $[2, 2, -2]$ two-bridge link where $\frac{\alpha}{\beta} = 8a_1 = \frac{8r}{s} > 0$ so that S_1^0 has boundary slope $4a_1$ on ∂V. We choose $\beta = 2s$, $\alpha = 16r$ so that F_{γ_1} is orientable.
we calculate the following:

- $\chi(S_1^0) = -\alpha - \beta = -2(8r + s)$,
- slope of ∂S_1^0 on T_W is $2\frac{\alpha}{\beta} = 2(\frac{8r}{s}) = \frac{16r}{s}$,
- $|\partial S_1^0 \cap T_K| = \gcd(2\beta, \alpha) = \gcd(4s, 16r) = 4\gcd(4, s)$,
- $|\partial S_1^0 \cap T_W| = \gcd(2\alpha, \beta) = \gcd(32r, 2s) = 2\gcd(16, s)$.

Jones surfaces arising from the maximum degree.

- $\chi(S_1^0) = -\alpha - \beta = -2(8r + s)$,
- slope of ∂S_1^0 on T_W is $2\frac{\alpha}{\beta} = 2(\frac{8r}{s}) = \frac{16r}{s}$,
- $|\partial S_1^0 \cap T_K| = \gcd(2\beta, \alpha) = \gcd(4s, 16r) = 4\gcd(4, s)$,
- $|\partial S_1^0 \cap T_W| = \gcd(2\alpha, \beta) = \gcd(32r, 2s) = 2\gcd(16, s)$.

The boundary of S consists of n copies of the boundary of S_1^0 on T_W, so we have

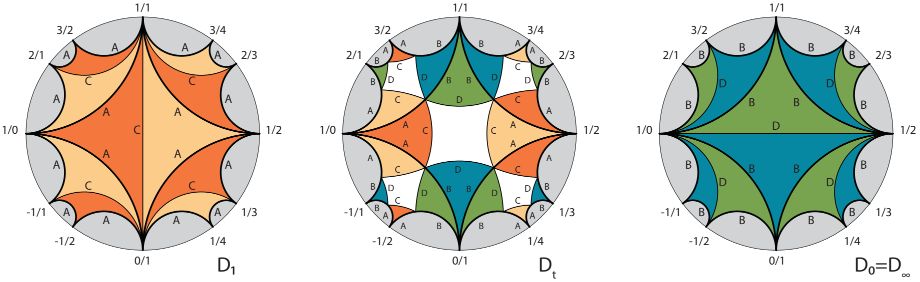
- $|\partial S| = n|\partial S_1^0 \cap T_W| = 2n\gcd(16, s)$.

Moreover, the boundary slope of S is the slope of ∂S_1^0 on T_W, and so this has denominator $\frac{s}{\gcd(16r,s)} = \frac{s}{\gcd(16,s)}$.

$$
\begin{aligned}
\frac{\chi(S)}{|\partial S| \cdot \frac{s}{\gcd(16,s)}} &= \frac{m\chi(S_K) + n\chi(S_1^0)}{2n\gcd(16,s) \cdot \frac{s}{\gcd(16,s)}} \\
&= \frac{2b_1 m|\partial S_K| \cdot \frac{s}{\gcd(4,s)} - 2n(8r+s)}{2ns} \\
&= \frac{8b_1 n\gcd(4,s) \cdot \frac{s}{\gcd(4,s)} - 2n(8r+s)}{2ns} \\
&= \frac{8b_1 ns - 2n(8r+s)}{2ns} \\
&= 4b_1 - 8r/s - 1 = 2(-4a_1 + 2b_1 - \frac{1}{2}) = 2b_W
\end{aligned}
$$

Algorithm

The diagram D_1 is the common Farey diagram. The diagram D_0 is obtained by switching the diagonal in each of the quadrilaterals. The diagram D_t is obtained by replacing these diagonals with inscribed quadrilaterals.

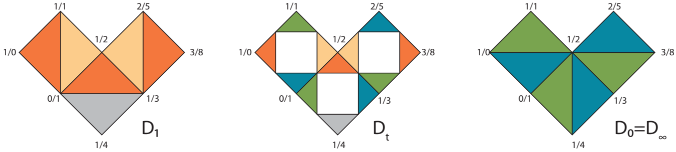


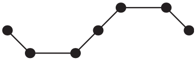
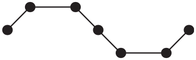
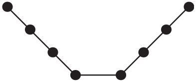
Floyd-Hatcher

For a two bridge link $\mathcal{L}_{p/q}$ (where q is even), a properly embedded essential surface in the exterior of the link is carried by one of finitely many branched surfaces associated to "minimal edge paths" in D_t from $1/0$ to p/q.

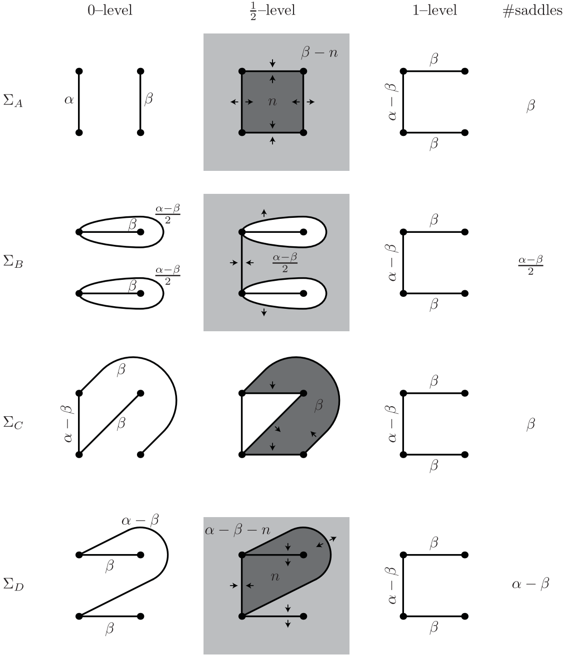
A *minimal edge path* in D_t is a consecutive sequence of edges of D_t such that the boundary of any face of D_t contains at most one edge of the path.

Example the Whitehead link $\mathcal{L}_{3/8}$



HS path	path picture	branch pattern	χ
γ_1		$ADAADA$	$-\alpha-\beta$
γ_2		$ADAADA$	$-\alpha-\beta$
γ_3		$ADAADA$	$-\alpha-\beta$
γ_5		$ADCDA$	$-\alpha$
γ_6		$ABBCBBA$	$-\alpha$

The four basic weighted branched surfaces



Data for essential surfaces in the exterior of $\mathcal{L}_{p/q}$

- the number of saddles gives the Euler characteristic χ.
- non-negative integral weights α and β indicate the algebraic (and geometric) intersection numbers of the surface with the meridians of the two components of $\mathcal{L}_{p/q}$. $\Rightarrow$ boundary slopes (Hoste-Shanahan)
- by a calculation in the homology of a torus, the $\gcd$ of the algebraic intersection numbers of the boundary of a surface with the meridian and longitudinal framing of a component of $\mathcal{L}_{p/q}$ produces the number of boundary components of the surface meeting that component of $\mathcal{L}_{p/q}$.

Data for essential surfaces in the exterior of the whitehead link $\mathcal{L}_{3/8}$

HS path	branch pattern	χ	boundary slopes $\beta>0$	number of boundary components $\beta>0$
γ_1	$ADAADA$	$-\alpha-\beta$	$(2\frac{\beta}{\alpha}, 2\frac{\alpha}{\beta})$	$(\gcd(2\beta,\alpha), \gcd(2\alpha,\beta))$
γ_2	$ADAADA$	$-\alpha-\beta$	$(0,0)$	(α,β)
γ_3	$ADAADA$	$-\alpha-\beta$	$(0,0)$	(α,β)
γ_5	$ADCDA$	$-\alpha$	$(-2\frac{\beta}{\alpha}, -2\frac{\alpha}{\beta}-2)$	$(\gcd(2\beta,\alpha), \gcd(2\alpha,\beta))$
γ_6	$ABBCBBA$	$-\alpha$	$(-4,-2)$	(α,β)
γ_5'	ACA	$-\alpha$	$(-4+\frac{X}{\beta}, -2-\frac{X}{\beta})$	$(\gcd(X,\alpha), \gcd(X,\beta))$