# Wigner's "continuous-spin" representations reconsidered

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(based on joint work with José M. Gracia-Bondía)

# Prologue: string-local field theory

Quantum fields can be built directly from positive-energy repns of the Poincaré group in the setting of Wigner's particle classification.

Standard treatments usually omit the so-called continuous-spin repns, that (a) so far have not been observed; and (b) cannot accomodate covariant "point-local" fields $\phi_r(x)$ [Yngvason, 1970].

But later, [Mund-Schroer-Yngvason, 2006] allowed for a string-local field $\phi_r(x,e)$, where $e^2 < 0$, localized in spacelike cones centered on "strings" or rays $\{x + te : t \geq 0\}$, and with good covariance properties:

$$U(a,\Lambda)\varphi_r(x,e)U^\dagger(a,\Lambda) = \varphi_s(\Lambda x + a, \Lambda e)\, D(\Lambda)^s_r\,.$$

String-local fields are available for all particle types; they "live on Hilbert space" (no indefinite metric); and satisfy string-locality: $[\varphi_r(x,e), \varphi_r(x',e')] = 0$ if $\{x + te\}, \{x' + t'e'\}$ are spacelike separated.

Recently, Rehren [2017] gave a construction of such quantum fields for continuous-spin representations, in the line of [MSY06].

Our aim here: to develop a "first-quantized" approach to such repns.

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# Origins: Wigner's particle classification

Wigner's 1939 paper classified the irreps of the Poincaré group $\mathcal{P}_+^\uparrow$ according to eigenstates of the 4-momentum $P_\mu$. This group has two Casimirs, $P^2$ and $W^2$, where $W^\mu := \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}P_\nu J_{\rho\sigma}$ is the Pauli-Lubański pseudovector. Note $(PW) \equiv P_\mu W^\mu = 0$, so $P^2 \geq 0$ implies $W^2 \leq 0$.

Disregarding $P^2 < 0$ and $P = 0$ repns, we are left with:

- $P^2 = m^2 > 0$ [so $W = -m^2 s(s+1)$], massive particles of spin $s$;
- $P^2 = 0$, $W^2 = 0$, "ordinary" massless particles;
- $P^2 = 0$, $W^2 = -\kappa^2 < 0$, the "last" particle species. These form two continuum families of repns (for $\kappa > 0$); there are bosonic and fermionic versions [Bargmann-Wigner, 1948].

The last case has often been dismissed as unobserved; indeed, no interaction with massive particles is known. But recently interest has revived, since it might contribute to the (largely unknown) material content of the universe.

I shall call the last case Wigner particles (WPs), for short.

# Wave equations for the WP (bosonic case)

As given by Wigner [1948], with $(x,w)$ or $(p,w)$ in $M^4 \times M^4$, these are:

$$\begin{aligned} \Box_x \Phi(x,w) &= 0; \quad \text{or} & p^2 \Phi(p,w) &= 0, \\ (w^2+\kappa^2)\Phi(x,w) &= 0; \quad \text{or} & (w^2+\kappa^2)\Phi(p,w) &= 0, \\ (w\partial_x)\Phi(x,w) &= 0; \quad \text{or} & (pw)\Phi(p,w) &= 0, \\ ((\partial_x\partial_w)+1)\Phi(x,w) &= 0; \quad \text{or} & ((p\partial_w)+i)\Phi(p,w) &= 0. \end{aligned}$$

The last comes from the form of $W^2$ acting on $(x,w)$-space:

$$\begin{aligned} (WW) &= -\tfrac{1}{2} J_{\nu\tau} J^{\nu\tau} P^2 + J_{\kappa\sigma} J^{\mu\sigma} P^\kappa P_\mu; \quad \text{with } P^2 = 0, \\ &= \kappa^2 (p\,\partial_w)^2 - (pw)^2 \Box_w + 2(pw)(p\,\partial_w)(w\,\partial_w) = -\kappa^2. \end{aligned}$$

which gives $(p\,\partial_w) = \pm i$ on the space of solutions. This integrates to $\Phi(p, w - \lambda p) = e^{\pm i\lambda} \Phi(p,w)$.

Schuster and Toro [2013-15] put $(p\,\partial_w)\Phi = 0$ instead, forcing $(pw) \neq 0$ and a different wave equation: $(pw)^2 \Box_w \Phi = \kappa^2 \Phi$.

# Classical elementary systems

Irreducible unitary repns of $\mathcal{P}_+^\uparrow$ match with coadjoint orbits (Kirillov).
For $m > 0$, the orbits are $\approx \mathbb{R}^6$ (for spin 0), or $\approx \mathbb{R}^6 \times \mathbb{S}^2$ (higher spins).

This even includes a Moyal formalism [Cariñena-GraciaB-JCV, 1990]: one can do relativistic QM on this platform.

The Lie-algebra generators $P^0, \mathbf{P}, \mathbf{L}, \mathbf{K}$ act as linear coordinates $p^0, \mathbf{p}, \mathbf{l}, \mathbf{k}$ on the orbits; commutators become Lie-Poisson brackets, $\{l^i, l^j\} = \varepsilon^{ij}{}_k\, l^k$, and so on.

Rotations $R_{\alpha\mathbf{m}} = \exp(\alpha\mathbf{m}\cdot\mathbf{L})$ fix $p^0$ and rotate $\mathbf{p}, \mathbf{l}, \mathbf{k}$ in the obvious way.
Here is the coadjoint action of the boosts $K_{\zeta\mathbf{n}} = \exp(\zeta\mathbf{n}\cdot\mathbf{K})$:

$$K_{\zeta\mathbf{n}} \triangleright p^0 = p^0 \cosh\zeta + \mathbf{n}\cdot\mathbf{p}\sinh\zeta,$$
$$K_{\zeta\mathbf{n}} \triangleright \mathbf{p} = \mathbf{p} + p^0\mathbf{n}\sinh\zeta + (\mathbf{n}\cdot\mathbf{p})\mathbf{n}(\cosh\zeta - 1),$$
$$K_{\zeta\mathbf{n}} \triangleright \mathbf{l} = \mathbf{l}\cosh\zeta + \mathbf{n}\times\mathbf{k}\sinh\zeta - (\mathbf{n}\cdot\mathbf{l})\mathbf{n}(\cosh\zeta - 1),$$
$$K_{\zeta\mathbf{n}} \triangleright \mathbf{k} = \mathbf{k}\cosh\zeta - \mathbf{n}\times\mathbf{l}\sinh\zeta - (\mathbf{n}\cdot\mathbf{k})\mathbf{n}(\cosh\zeta - 1).$$

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K_{\zeta\mathbf{n}} \triangleright p^0 &= p^0 \cosh\zeta + \mathbf{n}\cdot\mathbf{p}\sinh\zeta, \\
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# Moyal quantization: massive case

For $m > 0$, the $\mathbb{R}^6$'s in the orbits come from finding "canonical position coordinates" $q^i$ so that $\{q^i, p^j\} = \delta_{ij}$; the recipe is

$$\mathbf{q} := -\frac{\mathbf{k}}{p^0} + \frac{\mathbf{p} \times \mathbf{w}}{m p^0 (m + p^0)} = -\frac{\mathbf{k}}{p^0} + \frac{\mathbf{p} \times \mathbf{s}}{p^0 (m + p^0)}$$

where $\mathbf{s} := \mathbf{w}/m - w^0 \mathbf{p}/m(m + p^0)$ labels spin variables. Notice that $m\mathbf{s} \to \mathbf{w} - w^0 \mathbf{p}$ as $m \to 0$ with $m|\mathbf{s}|$ fixed.

When $|\mathbf{s}| > 0$, it is better to use $\mathbf{x} := \mathbf{q} - (\mathbf{p} \times \mathbf{s})/m(m + p^0)$. Then $u := (\mathbf{x}, \mathbf{p}, \mathbf{s}) \in \mathbb{R}^6 \times \mathbb{S}^2$ covariantly parametrizes the orbits.

For $j \in \frac{1}{2}\mathbb{N}$, the Moyal quantizer is a family of operators $\Omega^j(\mathbf{x}, \mathbf{p}, \mathbf{s})$ on $L^2(H_m^+, d\mu(\xi))$ defining a Weyl correspondence $W_A^j(u) := \mathrm{Tr}(A\,\Omega_j(u))$.

Its definition involves the reflections $M_p : \xi \mapsto 2(p\xi)p/(pp) - \xi$ on $H_m^+$ and the quantizer $\Delta^j(\mathbf{s})$ for the "fuzzy sphere" [JCV+GraciaB, 1989].

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# Coadjoint orbits for the WP

Recall the "canonical" $q^i$ on the massive orbits, given by

$$\mathbf{q} := -\frac{\mathbf{k}}{p^0} + \frac{\mathbf{p}\times\mathbf{w}}{mp^0(m+p^0)}, \quad \text{not so good when} \quad m \to 0.$$

Schwinger [1970] noted that in the "Pauli-Lubański-limit" $m \to 0$, $|\mathbf{s}| \to \infty$, with $m|\mathbf{s}|$ fixed:

$$m\mathbf{s} = \mathbf{w} - \frac{(\mathbf{w}\cdot\mathbf{p})}{p^0(m+p^0)}\,\mathbf{p} \longrightarrow \mathbf{w} - \frac{w^0}{p^0}\mathbf{p} =: \mathbf{t}$$

and suggested to replace $\mathbf{q}$ by a "position" vector $\mathbf{r}$, given by

$$-\frac{\mathbf{k}}{p^0} + \frac{\mathbf{p}\times\mathbf{w}}{(p^0)^2(m+p^0)} \longrightarrow -\frac{\mathbf{k}}{p^0} + \frac{\mathbf{p}\times\mathbf{w}}{(p^0)^3} = -\frac{\mathbf{k}}{p^0} + \frac{\mathbf{p}\times\mathbf{t}}{(p^0)^3} =: \mathbf{r}.$$

Now the helicity $\lambda := w^0/p^0$ satisfies $\{\lambda, \mathbf{r}\} = \mathbf{0}$. The price to pay is that $\{r^i, r^j\} = -\varepsilon^{ij}{}_k\, \lambda p^k (p^0)^{-3} \neq 0$.

We note also that $\{\lambda, \mathbf{t}\} = -\mathbf{p}/p^0 \times \mathbf{t}$ and $\{\lambda, \mathbf{p}/p^0 \times \mathbf{t}\} = \mathbf{t}$.

# Boosts and rotations: the gyroscope

We now focus on $\mathbf{w} = \lambda\mathbf{p} + \mathbf{t}$, where $\mathbf{t} \perp \mathbf{p}$ and $|\mathbf{t}| = \kappa$. The triple $(\mathbf{p}/|\mathbf{p}|, \mathbf{t}, \mathbf{p}/|\mathbf{p}| \times \mathbf{t})$ is an orthogonal frame in 3-space.

With $|\mathbf{p}| = p^0$, the boost $K_{\zeta\mathbf{n}}$ takes $\mathbf{p}/|\mathbf{p}|$ to another unit vector $\mathbf{p}'/|\mathbf{p}'|$ by a rotation $R_{\delta\mathbf{m}}$ with axis $\mathbf{m} \parallel \mathbf{p} \times \mathbf{n}$. Its angle $\delta$ is given by

$$\sin\delta = \frac{p^0 \sinh\zeta + (\mathbf{n}\cdot\mathbf{p})(\cosh\zeta - 1)}{p^0 p'^0}\, |\mathbf{p} \times \mathbf{n}|.$$

(This $\delta$ is the limiting angle of the Wigner rotation $B_{Kp}^{-1} K B_p$ as $m \to 0$.)

Pleasant surprise: the vectors $\mathbf{t}$ and $\mathbf{p}/|\mathbf{p}| \times \mathbf{t}$ undergo the same rotation: $K_{\zeta\mathbf{n}} \triangleright \mathbf{t} = R_{\delta\mathbf{m}}(\mathbf{t})$. Thus the frame rotates rigidly under boosts (and under rotations, too) [GraciaB-Lizzi-JCV-Vitale, 2018].

Orbit shape: $(\mathbf{r}, \mathbf{p}; \lambda, \theta) \in \mathbb{R}^3 \times (\mathbb{R} \times \mathbb{S}^2) \times (\mathbb{R} \times \mathbb{S}^1)$ where $\theta$ parametrizes the circle on which $\mathbf{t}$ and $\mathbf{p}/h \times \mathbf{t}$ live. (Say, $\mathbf{t} =: \mathbf{t}_1(\mathbf{p}) \cos\theta + \mathbf{t}_2(\mathbf{p}) \sin\theta$.) Moreover, $\{\lambda, \cos\theta\} = \sin\theta$, $\{\lambda, \sin\theta\} = -\cos\theta$.

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# Generators for a WP representation

We now return to (hermitian) generators of the Poincaré Lie algebra: impose $P^2 = 0$ and $W^2 = -\kappa^2$ to prepare a WP-type repn $U$ of $\mathcal{P}_+^\uparrow$.

Introduce $H := (\mathbf{P} \cdot \mathbf{L})(P^0)^{-1}$, so $W^0 = HP^0$.

Define $\mathbf{T} := \mathbf{W} - H\mathbf{P}$ and check that $[T^i, T^j] = 0$ and $\mathbf{T} \cdot \mathbf{T} = \kappa^2$.

Putting $\mathbf{Y} := \mathbf{P}(P^0)^{-1} \times \mathbf{T}$, one finds that $[H, Y^j] = iT^j$ and $[H, T^j] = -iY^j$, so that each pair $(Y^j, T^j)$ supplies ladder operators for $H$.

Under boosts (and rotations), the triple $(\mathbf{P}/P^0, \mathbf{T}, \mathbf{Y})$ still rotates gyroscopically.

The Schwinger position operators satisfy $[R^i, R^j] = -i\varepsilon^{ij}{}_k HP^k (P^0)^{-3}$, which already implies the nonlocality of the WP (as Schwinger noted).

Moreover, $[H, \mathbf{K}] = i\mathbf{T}(P^0)^{-1}$, so the helicity $H$ is not Lorentz-invariant in the WP representations.

# The one-particle Hilbert space

To display this representation in an invariant formalism, we may label states by pairs of 3-vectors $(\mathbf{p}, \mathbf{t})$, subject to $\mathbf{p} \neq \mathbf{0}$, $\mathbf{p} \perp \mathbf{t}$ and $|\mathbf{t}| = \kappa$.

The redundancy in $\mathbf{t}$ is removed as in [Bargmann-Wigner, 1948], by assigning an angle $\theta$ to the circle, $\mathbf{t} =: \mathbf{t}_1 \cos\theta + \mathbf{t}_2 \sin\theta$.

With $p^0 = |\mathbf{p}|$ and $\lambda := w^0/p^0$, we can simplify

$$\Phi(p, w) \equiv \Phi(\mathbf{p}, \lambda\mathbf{p} + \mathbf{t}) = e^{-i\lambda}\,\Phi(\mathbf{p}, \mathbf{t}) \equiv e^{-i\lambda}\,\Phi(\mathbf{p}, \theta)$$

and use the Lorentz-invariant scalar product

$$\langle \Phi \mid \Phi \rangle \propto \int \frac{d^3\mathbf{p}}{|\mathbf{p}|}\, d\theta\, |\Phi(\mathbf{p}, \theta)|^2.$$

On the space of solutions of the wave equations, the representation now has the scalar-like form:

$$U(a, \Lambda)\Phi(x, w) := \Phi(\Lambda^{-1}(x - a), \Lambda^{-1} w).$$

# What about the little group method?

The usual construction of a unirrep for the WP uses induction from the little group $E(2)$, generated by two "null rotations" and an ordinary rotation, replacing the $\mathbf{t}$-plane with an abstract plane.

Fix a 4-momentum $k = (|\mathbf{k}|, \mathbf{k})$ and take $\vec{\xi} \perp \mathbf{k}$ with $|\vec{\xi}| = \kappa$. We get rotation and boost generators [Lomont-Moses, 1962-67]:

$$\mathbf{L} \leftrightarrow -i\mathbf{p} \times \partial_{\mathbf{p}} + \frac{\mathbf{p} + |\mathbf{p}|\mathbf{k}}{|\mathbf{p}| + \mathbf{k}\cdot\mathbf{p}} \mathbf{S}\cdot\mathbf{k},$$

$$\mathbf{K} \leftrightarrow i|\mathbf{p}|\,\partial_{\mathbf{p}} - \frac{\mathbf{k}\times\mathbf{p}}{|\mathbf{p}| + \mathbf{k}\cdot\mathbf{p}} \mathbf{S}\cdot\mathbf{k} + \frac{\mathbf{p}}{|\mathbf{p}|^2} \times \left( \frac{\mathbf{p} + |\mathbf{p}|\mathbf{k}}{|\mathbf{p}| + \mathbf{k}\cdot\mathbf{p}} \times \vec{\xi} \right).$$

Contrast with, say, $\mathbf{L} \leftrightarrow -i\mathbf{p} \times \partial_{\mathbf{p}} - i\mathbf{w} \times \partial_{\mathbf{w}}$ for the invariant form.

A unitary transformation intertwines both representations:

$$\delta(|\vec{\xi}|^2 - \kappa^2)\,\delta(\vec{\xi}\cdot\mathbf{k})\,\psi(\mathbf{p},\vec{\xi}) := e^{iw^0/|\mathbf{p}|} \exp\!\left( i\alpha \frac{\mathbf{k}\times\mathbf{p}}{|\mathbf{k}\times\mathbf{p}|} \cdot \mathbf{L} \right) \Phi(p,w) \Big|_{\mathbf{w}=\vec{\xi}+w^0\mathbf{p}/|\mathbf{p}|}$$

with angle $\alpha$ such that $\cos\alpha = (\mathbf{k}\cdot\mathbf{p})/|\mathbf{p}|$.

# On second quantization for the WP: two remarks

The unitary transformation appears (without proof) in [Hirata, 1977], who also found a causal propagator of the form

$$\widetilde{D}(x,x';w,w') = \frac{\delta(w^2+\kappa^2)}{(2\pi)^3} \int d^3\mathbf{p}\, \frac{\sin|\mathbf{p}|(t-t')}{|\mathbf{p}|}\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \times \delta(pw)\,\delta^3\Big(|\mathbf{p}|(\mathbf{w}-\mathbf{w}') - (w^0 - w'^0)\mathbf{p}\Big)\, e^{i(w^0-w'^0)/|\mathbf{p}|}.$$

This $\widetilde{D}$ is Lorentz-invariant and satisfies the Wigner equations.

(Here $\Phi(x,w)$ depends on the extra $w$, so this does not contradict Yngvason's theorem on nonlocality of quantum fields for the WP.)

We know that a quantized field for a WP could be string-local, so we can try to find a good set of intertwiners. Indeed, this has already been done in [Rehren, 2017], but with a different starting point, leading to a stress-energy-momentum tensor for the WP, as a quadratic form.

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