Painlevé equations from Nakajima-Yoshioka blow-up relations

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Painlevé III_3 equation and its q-deformation

Object of our interest: Painlevé III_3 equation, its q-deformation and their solutions.

- Toda-like form of these equations is a two bilinear equations on two functions: τ and τ_1. It is symmetric under $\tau \leftrightarrow \tau_1$.
- For the Painlevé III_3

$$\begin{aligned} D^2_{[\log z]}(\tau,\tau) &= -2z^{1/2}\tau_1^2 \\ D^2_{[\log z]}(\tau_1,\tau_1) &= -2z^{1/2}\tau^2 \end{aligned} \tag{1}$$

  where second Hirota differential $D^2_{[\log z]}(\tau,\tau) = 2\tau''\tau - \tau'^2$, $f' = z\frac{df}{dz}$

- q-deformation

$$\begin{aligned} \overline{\tau}\underline{\tau} &= \tau^2 - z^{1/2}\tau_1^2 \\ \overline{\tau_1}\underline{\tau_1} &= \tau_1^2 - z^{1/2}\tau^2 \end{aligned} \tag{2}$$

  where $\overline{\tau(z)} = \tau(qz), \underline{\tau(z)} = \tau(q^{-1}z)$.

Gamayun-Iorgov-Lisovyy τ function

Gamayun-Iorgov-Lisovyy in 2012-2013 proposed power series representation for the τ function of the (continuous) Painlevé equations. τ function — Fourier transformation of Nekrasov functions

$$\tau_j(\sigma, s|z) = \sum_{n\in\mathbb{Z}+j/2} s^n \mathcal{Z}(\sigma+n|z) \tag{3}$$

- $\mathcal{Z}(\sigma|z)$ is a certain Nekrasov function which depends on the Painlevé equation we take. It is a power series of z depending on vacuum expectation value σ and possibly other parameters.
- s and σ play role of the integration constants of the Painlevé equation.
- We take $j=0$ for τ and $j=1$ for τ_1.

Approach to solve bilinear equations on τ functions

- We have some bilinear equation on some τ function, schematically

$$\langle \tau, \tau \rangle = 0 \tag{4}$$

- We take the Gamayun-Iorgov-Lisovyy formula as the ansatz with some function $\mathcal{Z}(\sigma|z)$.
- We collect terms with the power s^m. Due to the structure of the ansatz, the relations with s^m is equivalent to the relations with s^{m-1}

$$\langle \tau, \tau \rangle = 0, \quad \tau = \sum_{n\in\mathbb{Z}} s^n \mathcal{Z}(\sigma+n|z) \Leftarrow \langle \tau, \tau \rangle|_{s^m} = 0 \Leftrightarrow \langle \tau, \tau \rangle|_{s^0} = 0 \tag{5}$$

- s^0 relations is some relation on $\mathcal{Z}(\sigma|z)$. In our cases $\mathcal{Z}(\sigma|z)$ is some Nekrasov function or via the AGT some conformal block.

Instanton partition functions

- Nekrasov instanton partition function for pure gauge $U(r)$ YM is defined as the equivariant volume of the instanton moduli space

$$\mathcal{Z}(\epsilon_1, \epsilon_2, \vec{a}; z) = \sum_{N=0}^{+\infty} z^N \int_{M(r,n)} 1 \tag{6}$$

- This integral localizes on the fixed points of action of the $r+2$-dimensional torus. Fixed points are labeled by r-tuple of the Young diagrams.
- Example for the 4D $SU(2)$ case

$$\begin{aligned} \mathcal{Z}_{inst}(a_1, a_2; \epsilon_1, \epsilon_2 | z) &= \sum_{\lambda^{(1)},\lambda^{(2)}} \frac{z^{|\lambda^{(1)}|+|\lambda^{(2)}|}}{\prod_{i,j=1}^{2} \mathrm{N}_{\lambda^{(i)},\lambda^{(j)}}(a_i - a_j; \epsilon_1, \epsilon_2)}, \\ \mathrm{N}_{\lambda,\mu}(a; \epsilon_1, \epsilon_2) &= \\ = \prod_{s\in\lambda}(a - \epsilon_2(a_\mu(s)+1) + \epsilon_1 l_\lambda(s)) &\prod_{s\in\mu}(a + \epsilon_2 a_\lambda(s) - \epsilon_1(l_\mu(s)+1)) \end{aligned} \tag{7}$$

Structure of the Nekrasov function

$$\mathcal{Z} = \mathcal{Z}_{cl}\mathcal{Z}_{1-loop}\mathcal{Z}_{inst}, \tag{8}$$

- $\mathcal{Z}_{inst}$ is given by the Nekrasov formula. Case 5D differs from the case 4D in the Nekrasov formula by $a \mapsto 1 - q^a$.
- Classical part $\mathcal{Z}_{cl} = z^{\Delta(\sigma)}$.
- $\mathcal{Z}_{1-loop}$ is given by the double gamma functions with periods $\epsilon_{1,2}$ but when $\epsilon_1/\epsilon_2 \in \mathbb{Z}$ it is expressed in G function (or its q-deformation by q-Pochhammers in 5D case)

$$\mathcal{Z}_{1-loop,\epsilon_1=-\epsilon_2} = \frac{1}{\mathsf{G}(1-2\sigma)\mathsf{G}(1+2\sigma)}, \quad \mathcal{Z}_{1-loop,\epsilon_1=-2\epsilon_2} = \mathcal{Z}^{1/2}_{1-loop,\epsilon_1=-\epsilon_2} \tag{9}$$

Nekrasov functions and solutions of Painlevé equations

Different choice of $\mathcal{Z}$:

- 4D $SU(2)$ Nekrasov function with $\epsilon_1 + \epsilon_2 = 0$ (or Vir $c = 1$ conformal block) give the solution of continuous Painlevé equations $PVI, V, III's$ (Gamayun-Iorgov-Lisovyy, 2012-2013) — proved by different methods.
- 5D $SU(2)$ Nekrasov function with $\epsilon_1 + \epsilon_2 = 0$ — q-deformed Painlevé III_3 equation (Bershtein-S.,2016) and q-deformed Painlevé VI equation (Jimbo-Nagoya-Sakai, 2017)
- Other choices lead to generalization for the isomonodromic problems of higher rank, larger number of punctures, generalization of Toda-like equations with larger number of nodes, quantization of the τ function.... (Bershtein, Gavrylenko, Marshakov, Iorgov, Lisovyy 2015-2018)

Bilinear relations on 5D $SU(2)$ Nekrasov function

In case of q-deformed Painlevé III_3 we use conjectural bilinear relation on 5D Nekrasov functions

$$\sum_{2n\in\mathbb{Z}} \mathcal{Z}(\sigma+n|q^{-1}z)\mathcal{Z}(\sigma-n|qz) = (1-z^{1/2})\sum_{2n\in\mathbb{Z}} \mathcal{Z}(\sigma+n|z),\mathcal{Z}(\sigma-n|z) \quad (10)$$

One of the aims of this talk is to present the proof of this relation which before was checked numerically up to the power z^{12}.
The continuous limit $z\mapsto R^4 z, q=e^R, R\to 0$ give us

$$\sum_{2n\in\mathbb{Z}} D^2(\mathcal{Z}(\sigma+n|z),\mathcal{Z}(\sigma-n|z)) = -z^{1/2}\sum_{2n\in\mathbb{Z}} \mathcal{Z}(\sigma+n|z),\mathcal{Z}(\sigma-n|z) \quad (11)$$

We have proved these relations by the representation theory of Super Virasoro algebra when we proved the result of Gamayun-Iorgov-Lisovyy for the continuous Painlevé equation.

Nakajima-Yoshioka relations

There are blow-up relations on 4D and 5D partition functions [Nakajima, Yoshioka 2003,2005]. They express instanton partition function on $\widehat{\mathbb{C}^2}$ — $\mathbb{C}^2$ blowed up in the point as a bilinear relation on $\mathbb{C}^2$ instanton partition function

$$\mathcal{Z}_{\widehat{\mathbb{C}^2}}(a|\epsilon_1,\epsilon_2|z) = \sum_{n\in\mathbb{Z}} \mathcal{Z}_{\mathbb{C}^2}(a+\epsilon_1 n|\epsilon_1,\epsilon_2-\epsilon_1|z)\mathcal{Z}_{\mathbb{C}^2}(a+\epsilon_2 n|\epsilon_1-\epsilon_2,\epsilon_2|z) \quad (12)$$

and

$$\mathcal{Z}_{\widehat{\mathbb{C}^2}}(a|\epsilon_1,\epsilon_2|z) = \mathcal{Z}_{\mathbb{C}^2}(a|\epsilon_1,\epsilon_2|z) \quad (13)$$

There are also differential (for 4D) and q-difference (for 5D) Nakajima-Yoshioka relations.

Nakajima-Yoshioka relations: $c = -2$ τ function

Take particular case $\epsilon_1 + \epsilon_2 = 0$ in Nakajima-Yoshioka relations . Then in CFT terms $c = 1$ partition function is a bilinear combination of $c = -2$

$$\mathcal{Z}_{c=1}(\sigma|z) = \sum_{n\in\mathbb{Z}} \mathcal{Z}_{c=-2}(\sigma+n|z/4)\mathcal{Z}_{c=-2}(\sigma-n|z/4) \tag{14}$$

Then it is natural to make Fourier transformation

$$\tau(\sigma,s|z) = \tau^-(\sigma,s|z)\tau^+(\sigma,s|z) \tag{15}$$

where

$$\tau^\pm(\sigma,s|z) = \sum_{n\in\mathbb{Z}} s^{n/2}(\pm i)^{n^2}\mathcal{Z}_{c=-2}(\sigma+n|z/4) \tag{16}$$

Differential Nakajima-Yoshioka relations

$$\begin{aligned} D^1_{[\log z]}(\tau^-,\tau^+) &= z^{1/4}\tau_1, & D^2_{[\log z]}(\tau^-,\tau^+) &= 0 \\ D^3_{[\log z]}(\tau^-,\tau^+) &= z^{1/4}\tau_1', & D^4_{[\log z]}(\tau^-,\tau^+) &= -2z\tau \end{aligned} \tag{17}$$

The PIII_3 equation follow from these equations. Namely, first and second equations imply that

$$z\frac{d}{dz}\tau^{\pm} = \frac{1}{2}(\zeta \pm \zeta')\tau^{\pm} \tag{18}$$

where $\zeta = \tau'/\tau$ is a Hamiltonian of the PIII_3 equation. From the fourth equation

$$(\zeta'' - \zeta')^2 = 4\zeta'^2(\zeta - \zeta') - 4z\zeta' \tag{19}$$

which is Hamiltonian form of PIII_3 equation.

Toda-like form from Nakajima-Yoshioka relations

q-difference Nakajima-Yoshioka relations

$$\overline{\tau^+}\underline{\tau^-} - \underline{\tau^+}\overline{\tau^-} = -2z^{1/4}\tau_1,$$
$$\overline{\tau^+}\underline{\tau^-} + \underline{\tau^+}\overline{\tau^-} = 2\tau \tag{20}$$

Proposition

Take (20) and $\tau = \tau^+\tau^-$. Then τ and τ_1 satisfy Toda-like equation

$$\overline{\tau}\underline{\tau} = \tau^2 - z^{1/2}\tau_1^2 \tag{21}$$

Proof and consequences

Proof.

Proof is extremely elementary. We substitute τ_1 and τ in different ways

$$\overline{\tau^+}\,\overline{\tau^-}\,\underline{\tau^+}\,\underline{\tau^-} = \frac{1}{4}(\overline{\tau^+}\,\underline{\tau^-} + \underline{\tau^+}\,\overline{\tau^-})^2 - \frac{1}{4}(\overline{\tau^+}\,\underline{\tau^-} - \underline{\tau^+}\,\overline{\tau^-})^2 \tag{22}$$

□

- Nakajima-Yoshioka relations are proven, so we obtain the proof of the conjectured bilinear relation (10) in q-case automatically.
- There is, of course, differential analogue of this Proposition.

$c = -2$ τ function: motivations 1

- Riemann-Hilbert problem: [Iorgov, Lisovyy, Teschner, 2014] give naturaly relate $\tau_{c=1}$ to the Riemann-Hilbert problem. Their arguments works for the case $b^2 \in \mathbb{Z}$, in particular for $c = -2$.
- Central charge of symplectic fermions is $c = -2$. We expect that $\tau_{c=-2}$ is a conformal block for the symplectic fermions.
- Classical conformal block $c = \infty$ is related to the Painlevé equations [Litvinov, Lukyanov, Nekrasov, Zamolodchikov, 2013]. There is a relation between $\tau_{c=-2}$ and classical conformal blocks.
- For special resonance values of parameters $\tau_{c=1}$ is expressed by the determinant of hypergeometric functions [Morozov, Mironov, 2017]. Similarly $\tau_{c=-2}$ is a Pfaffian.

$c = -2$ τ function: motivations 2

[Bonelli-Grassi-Tanzini, 2017]:

$$\tau(\kappa, \hbar, \xi) = Z_{CS}(\hbar, \xi) \det(1 + \kappa \rho_{\mathbb{P}^1 \times \mathbb{P}^1}), \tag{23}$$

where ρ is inverse operator to the Hamiltonian of the relativistic Toda chain

$$\rho_{\mathbb{P}^1 \times \mathbb{P}^1} = (e^p + e^{-p} + e^x + m_{\mathbb{P}^1 \times \mathbb{P}^1} e^{-x})^{-1} \tag{24}$$

This determinant is the grand canonical partition function for topological strings on local $\mathbb{P}^1 \times \mathbb{P}^1$. Spectral determinant for the special value of z is a generating function for the ABJ partition functions.

$$\overline{\tau^+}\underline{\tau^-} + \underline{\tau^+}\overline{\tau^-} = \tau = \tau^+\tau^- \tag{25}$$

are "quantum Wronskian" relations in ABJ theory [Grassi-Hatsuda-Marino, 2014]

$c = -2$ τ function: cluster structure

q-deformed $c = -2$ τ function admits cluster structure as the $c = 1$ τ function does [Bershtein, Gavrylenko, Marshakov, 2017]

$$\overline{\tau_0^+} = \frac{\tau_0^+ \tau_0^- - z^{1/4} \tau_1^+ \tau_1^-}{\underline{\tau_0^-}}$$
$$\overline{\tau_0^-} = \frac{\tau_0^+ \tau_0^- + z^{1/4} \tau_1^+ \tau_1^-}{\underline{\tau_0^+}} \tag{26}$$

The quiver is surprisingly just the same as for the q-Painlevé VI equation.

$c = -2$ τ function: Chern-Simons terms generalization

- In the work Bershtein, Gavrylenko, Marshakov, 2018 the generalizations of Toda-like equations were considered. The generalizations were in two directions – the number of nodes bigger than 2 and Toda-like equations corresponding to the Nekrasov functions modified by the Chern-Simons term.
- We at the moment failed to obtain Toda-like equations from Nakajima-Yoshioka relations for the number of nodes larger then 2 but for the case of 2 nodes we have obtained Chern-Simons generalization.
- Each summand of instanton partition function obtain a multiplier in power k

$$\mathsf{T}_\lambda = \prod_{(i,j)\in\lambda} u^{-1} q_1^{1-i} q_2^{1-j} \tag{27}$$

- The level of additional Chern-Simons is $0 \leq k \leq r$ — only in this case instanton partition function converge.
- $k = 2$ is equivalent to the $k = 0$ and $k = 1$ correspond to the Painlevé $A_7^{(1)}$ equation.

Algebraic solutions and log limit of $c = -2$ τ function

As it is for $c = 1$ τ function there exist limit $\sigma \to 0$ and "algebraic" solution for the $\sigma = 1/4, s = \pm 1$

- It is known that $\tau(1/4, \pm 1|z) = z^{1/16} e^{\mp\sqrt{z}}$. One can find that

$$\tau^{+} = e^{2iz^{1/4}} z^{1/32} e^{2\sqrt{z}}, \quad \tau^{-} = e^{2iz^{1/4}} z^{1/32} e^{2\sqrt{z}} \tag{28}$$

- Logarithmic limit for $c = 1$ is $s = e^{2\Omega\sigma}, \sigma \to 0$. It also could be applied for the $c = -2$ τ function

Thank you for the attention!

Conformal blocks

- We have graded Lie algebra (for example, Virasoro algebra) $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$.
- We take highest weight vector $|v_\lambda\rangle$, s.t. $U(\mathfrak{n}^+)$ act on it by zero and $U(\mathfrak{h})$ act as on eigenvector.
- Verma module $U(\mathfrak{n})^-|v_\lambda\rangle$
- Whittaker vector in the Verma module

$$|W(z)\rangle = \sum_{N=0}^{+\infty} z^N |N\rangle, \quad deg(|N\rangle) = N \tag{29}$$

s.t.

$$g|W(z)\rangle = \beta_g z^{deg(g)} |W(z)\rangle \tag{30}$$

- Conformal block

$$\mathcal{Z}(z) = \langle W(1)|W(z)\rangle \tag{31}$$

Representation-theoretic interpretation

Nakajima-Yoshioka blow-up relations on 4D Nekrasov functions have representation-theoretic interpretation (Bershtein, Feigin, Litvinov, 2013).

- 4D Nekrasov function with $\epsilon_1 + \epsilon_2 = 0$ correspond via AGT correspondence to the $c = 1$ 4-point Virasoro conformal block.
- Introduce vertex operator algebra *Urod* as a sum of Heisenberg algebra Fock modules $\bigoplus_{k\in\mathbb{Z}} F_{k\sqrt{2}}$ but with modified stress-energy tensor.
- There are $\mathrm{Vir} \oplus \mathrm{Vir}$ subalgebra with $b_1^2 + b_2^{-2} = -1$ in the $U(Urod \otimes \mathrm{Vir})$, moreover there is a decomposition of the Verma module

$$U_1 \otimes \mathbb{L}_{P,b} = \bigoplus_{k\in\mathbb{Z}} \mathbb{L}_{P_1+kb_1,b_1} \otimes \mathbb{L}_{P_2+kb_2^{-1},b_2} \tag{32}$$

- Take Whittaker vector $v_{1/\sqrt{2}} \otimes |W(z)\rangle$ in l.h.s. Then it decompose into sum of $\mathrm{Vir} \oplus \mathrm{Vir}$ Whittaker vectors in r.h.s. Squaring this relation we obtain Nakajima-Yoshioka blow-up relation.