Cluster integrable systems, deautonomization and q-difference isomonodromic problem

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Tau Functions of Integrable Systems and Their Applications

Banff, September, 2018

based on

Cluster integrable systems and q-Painlevé equations, JHEP 02 (2018) 077, arXiv:1711.02063

Cluster Toda chains and Nekrasov functions, to appear in L.D.Faddeev volume of Theor. & Math. Phys., arXiv:1804.10145

with Misha Bershtein & Pasha Gavrylenko (their talks!)

some development ("Spin chain case" etc) yet to appear,
also with Kolya Semenyakin ...

Tau Functions of Integrable Systems

NOT quite true ...

Actually:

- only "traces of integrability" – from CLUSTER integrable systems;
- lead to q-difference equations: more simple than differential;
- discrete flows, but from "normal" Hamiltonian systems;
- the main issue – SOLUTIONS (in the following talk of Misha Bershtein), coming from 5d supersymmetric gauge theories and topological strings...

Integrable systems – too simple for that ...

DEAUTONOMIZATION!

Integrability

Many faces of INTEGRABILITY:

- Dubrovin-Krichever-Novikov: algebraic curve Σ and two meromorphic differentials (dE, dW) with *fixed* periods;
- Flat co-ordinates: $a = \oint_A EdW$

$$\frac{\partial \mathcal{F}}{\partial a} = \oint_B EdW \tag{1}$$

- *Integrability* ensured by the Riemann bilinear identities (e.g. symmetricity of the period matrix $T_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}$ of Σ).

Application

Seiberg-Witten integrable systems (from SUSY gauge theories):

- Pure gauge theories $\equiv$ Toda chains with $\mathfrak{g} = \mathrm{Lie}(G)$ of the gauge group;
- Lax representation: $L \in \widehat{\mathfrak{g}} \otimes K(\Sigma)$, algebraic curve

$$\det(L(\mu) - \lambda) = 0 \tag{2}$$

  with differentials of two functions $E = \lambda$, $W = \log\mu$, i.e. $\Sigma \subset \mathbb{C} \times \mathbb{C}^\times$.
- "Relativization" (4d $\to$ 5d) or "trigonometrization": symmetric situation $E = \log\lambda$, $W = \log\mu$ for $\Sigma \subset \mathbb{C}^\times \times \mathbb{C}^\times$. Lax operator g "$= \exp(L)$" $\in \widehat{G}$: co-extended loop group.

Cluster integrable system

a la Goncharov-Kenyon and/or Fock-AM:

- Defined by *any* convex NP $\Delta \subset \mathbb{Z}^2 \subset \mathbb{R}^2$ for a curve $\Sigma \subset \mathbb{C}^\times \times \mathbb{C}^\times$

$$f_\Delta(\lambda, \mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0. \qquad (3)$$

- Realized on a Poisson X-cluster variety $\mathcal{X}$, $\dim \mathcal{X} = 2\mathrm{Area}(\Delta)$. Poisson structure

$$\{x_i, x_j\} = \epsilon_{ij} x_i x_j, \qquad \{x_i\} \in \left(\mathbb{C}^\times\right)^{2\mathrm{Area}(\Delta)}. \qquad (4)$$

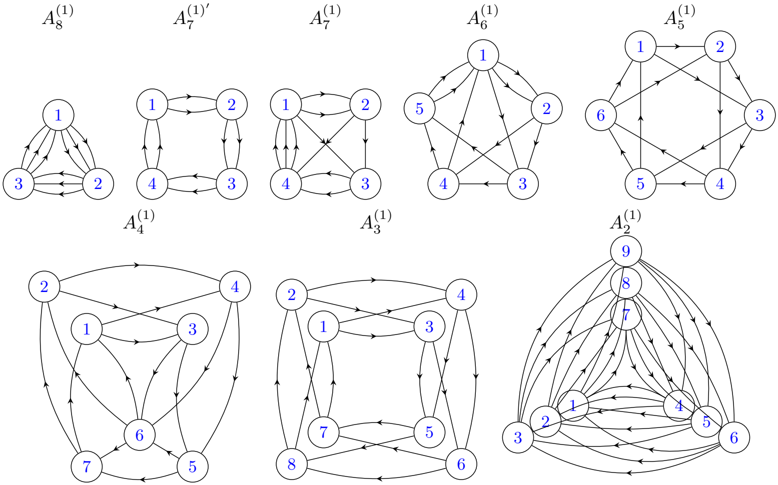
is encoded in a quiver $\mathcal{Q}$, with $\epsilon_{ij} = \#\mathrm{arrows}(i \to j)$.

- Integrability: Pick's formula

$$2\mathrm{Area}(\Delta) - 1 = (B - 3) + 2g \qquad (5)$$

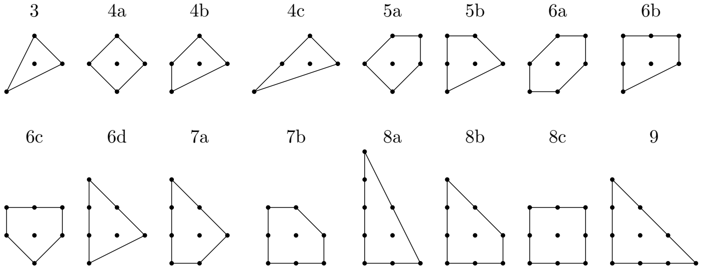
Quivers

$\mathcal{Q}$ of the "Painlevé cluster varieties" (with their q-Painlevé names), come from



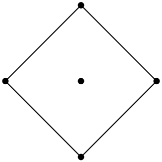
Newton polygons

Δ with a single internal point and $3 \leq B \leq 9$ boundary points:



Here Σ: $f_\Delta(\lambda, \mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0$ is always a torus $g = 1$.

Example



NP (up to $SA(2,\mathbb{Z})$-tranform):

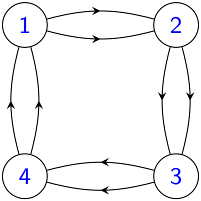
$$f_\Delta(\lambda,\mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = \lambda + \frac{1}{\lambda} + \mu + \frac{z}{\mu} + u = 0 \tag{6}$$

spectral curve for relativistic affine 2-particle Toda chain at $H(\vec{x}) = u$ (5d pure $SU(2)$ gauge theory).

Remark: renormalizations of λ, μ and f_Δ fix 3 of coefficients $\{f_{a,b}\}$ in the equation.

Example

X-cluster Poisson variety with (mutation class of) quiver $\mathcal{Q}$:



encoding logarithmically constant Poisson bracket

$$\{x_i, x_j\} = \epsilon_{ij} x_i x_j, \qquad i, j = 1, \dots, |\mathcal{Q}| \tag{7}$$

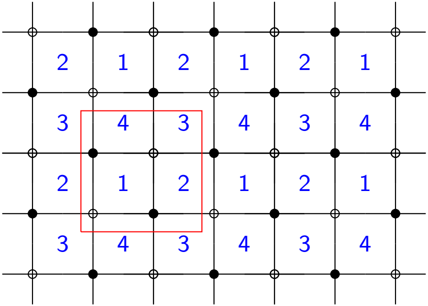
with the skew-symmetric matrix

$$\epsilon_{ij} = -\epsilon_{ji} = \#\mathrm{arrows}\ (i \to j) = \pm 2 \tag{8}$$

Obviously $q = x_1 x_2 x_3 x_4$ and $z = x_1 x_3$ are in the center of Poisson algebra.

Example

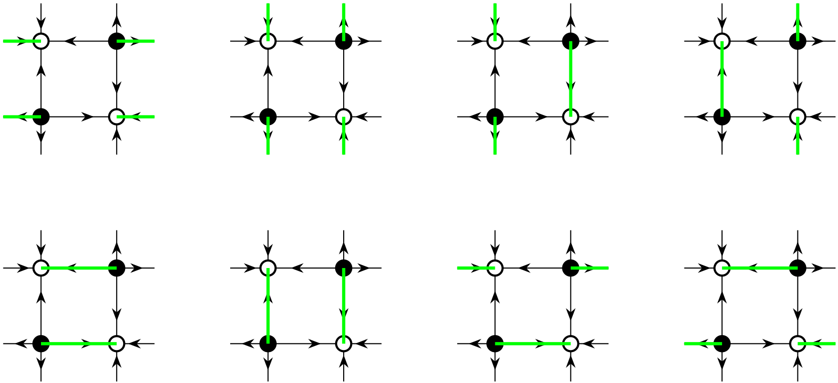
GK definition: the dimer partition function on a bipartite graph



gives rise (for $q = 1!$) to an integrable system with a 5d SW spectral curve $Z_{\rm dimer} \sim f_\Delta = \lambda + \frac{1}{\lambda} + \mu + \frac{z}{\mu} + H(\vec{x})$.

Example

Dimer configurations:



bipartite graph ⇒ chains ⇒ loops

Example

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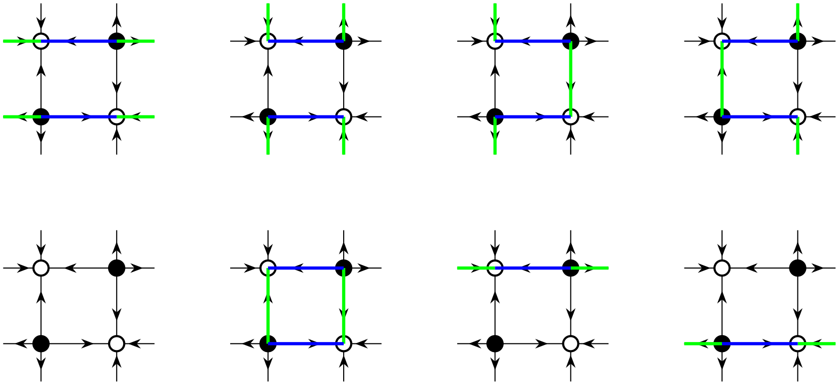


bipartite graph ⇒ chains ⇒ loops

Partition function and curve

$A \in GL(1)$ connection on a graph. Weights of configuration D:

$$W(D) = (-)^{Q(D)} \prod_{edges \in D} A_{edge}$$

$D - D_0$ is a combination of closed loops: $\partial(D - D_0) = 0$.

Parametrization of the connection (integrals over elementary closed loops):

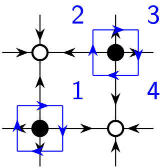
$$\prod_{e \in \partial Face_i} A_e = x_i, \quad \prod_{e \in A-cycle} A_e = \lambda, \quad \prod_{e \in B-cycle} A_e = \mu$$

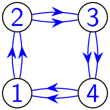
Important: $q = \prod_i x_i = \prod_{e \in \partial \mathbb{T}^2} A_e = 1$, since $\partial \mathbb{T}^2 = 0$.

Partition function:

$$W(D_0)^{-1} \sum W(D) = Z_{\text{dimer}}(\lambda, \mu; \vec{x}) = \sum_{(a,b) \in \Delta} \lambda^a \mu^b f_{a,b}(\vec{x})$$

Poisson structure

Poisson quiver $\mathcal{Q}$: 



is defined by

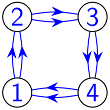
- intersection form in $H_1(\Sigma)$ of dual surface Σ to $\mathbb{T}^2$: Darboux co-ordinates;
- Poisson quiver $\mathcal{Q}$: cluster variables.

Involution: $\{f_{a,b}\} \to \{\vec{z}, \vec{H}\}$, so that

$$\{\vec{z}, x_i\} = 0, \qquad \{H_I, H_J\} = 0 \tag{9}$$

Mutations

Cluster mutations on X-cluster variety:



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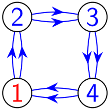

Mutation μ_1

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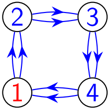

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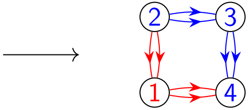

Reverse all incoming and outgoing arrows

$x_1' = 1/x_1$

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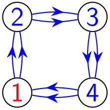

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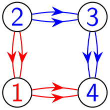

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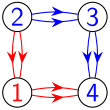

Complete cycles through mutation vertex

$x_4' = x_4(1+x_1)^2$

$x_2' = x_2(1+1/x_1)^{-2}$

Mutations

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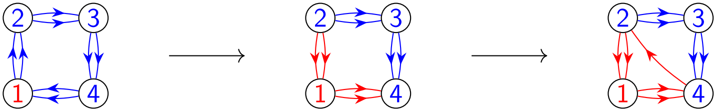

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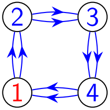

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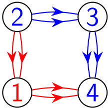

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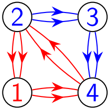

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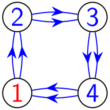

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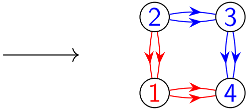

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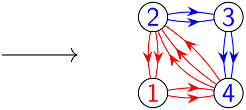

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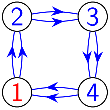

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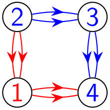

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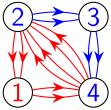

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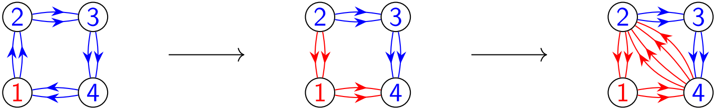

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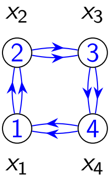
Formulas: $\mu_j : \epsilon_{ik} \mapsto -\epsilon_{ik}$, if $i = j$ or $k = j$, $\quad \epsilon_{ik} \mapsto \epsilon_{ik} + \frac{\epsilon_{ij}|\epsilon_{jk}| + \epsilon_{jk}|\epsilon_{ij}|}{2}$ otherwise.

$$\mu_j : x_j \mapsto x_j^{-1}, \qquad x_i \mapsto x_i \left(1 + x_j^{\mathrm{sgn}\,\epsilon_{ij}}\right)^{\epsilon_{ij}}, \quad i \neq j. \qquad \{x_i', x_k'\} = \epsilon_{ik}' x_i' x_k'$$

Cluster automorphisms

All combinations of mutations, permutations of vertices and simultaneous inversions of edges, that preserve quiver $\mathcal{G}_{\mathcal{Q}} \supset \mathcal{G}_{\Delta}$ (discrete flows of IS).
Example – the flow T:



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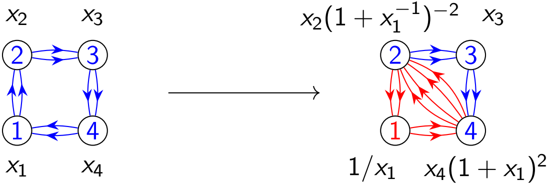

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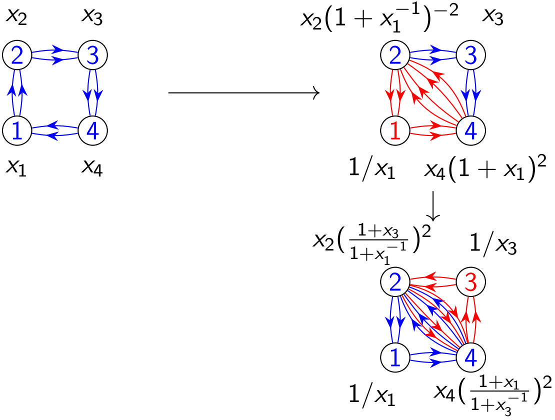

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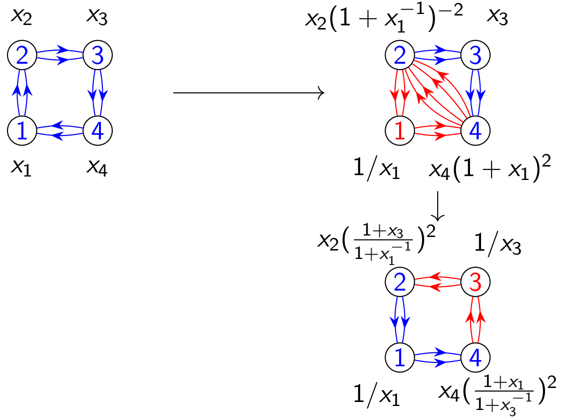

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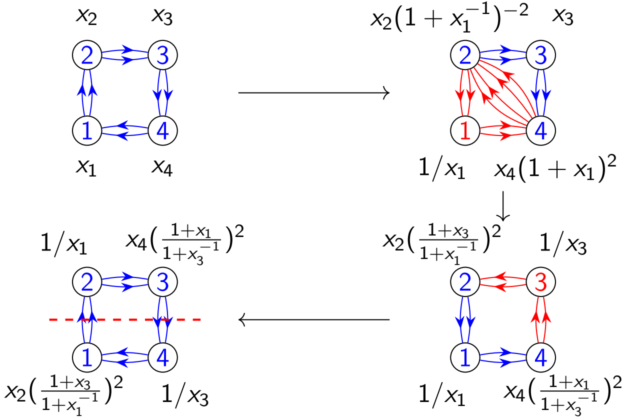


Deautonomization

For $q = 1$ the flow T

$$T : (x_1, x_2, x_3, x_4) \mapsto \left(x_2 (\frac{1 + x_3}{1 + x_1^{-1}})^2, x_1^{-1}, x_4 (\frac{1 + x_1}{1 + x_3^{-1}})^2, x_3^{-1} \right)$$

preserves Hamiltonian $H = \sqrt{x_1 x_2} + \frac{1}{\sqrt{x_1 x_2}} + \sqrt{\frac{x_1}{x_2}} + z\sqrt{\frac{x_2}{x_1}}$.

Let $x_1 x_2 x_3 x_4 = q \neq 1$ (*no* integrable system!)

$$T : (x_1, x_2, z, q) \mapsto \left(x_2 (\frac{x_1 + z}{x_1 + 1})^2, x_1^{-1}, qz, q \right)$$

Casimir z as "time" $x_i = x_i(z)$, $T : x_i(z) \mapsto x_i(qz)$, satisfying

$$x_1(qz) x_1(q^{-1} z) = \left(\frac{x_1(z) + z}{x_1(z) + 1} \right)^2$$

or q-Painlevé III_3 equation $P(A_7^{(1)'})$.

Quantization

Remark:

- In addition to non-autonomous parameter q one may add quantum deformation p:

$$\hat{x}_i \hat{x}_j = p^{-2\epsilon_{ij}} \hat{x}_j \hat{x}_i$$

  just *quantizing* the X-cluster variety.

- Quantum mutations

$$\mu_j: \quad \hat{x}_j \mapsto \hat{x}_j^{-1}, \quad \hat{x}_i^{1/|\epsilon_{ij}|} \mapsto \hat{x}_i^{1/|\epsilon_{ij}|} \left(1 + p\hat{x}_j^{\operatorname{sgn} \epsilon_{ij}}\right)^{\operatorname{sgn} \epsilon_{ij}}, \; i \neq j$$

- Quantum q-Painlevé equations, e.g. quantum q-Painlevé III_3:

$$\begin{cases} \hat{x}_1(q^{-1}z)^{1/2}\, \hat{x}_1(qz)^{1/2} = \dfrac{\hat{x}_1(z) + pz}{\hat{x}_1(z) + p}, \\ \hat{x}_1(Z)\hat{x}_1(q^{-1}z) = p^4 \hat{x}_1(q^{-1}z)\hat{x}_1(qz). \end{cases}$$

- Important since SOLUTION still exists!

Tau-functions

For the tau-functions $x_1(z) = z^{1/2}\frac{\tau_3(z)^2}{\tau_1(z)^2}$ one gets bilinear (non-autonomous!) *Hirota equations*

$$\tau_1(qz)\tau_1(q^{-1}z) = \tau_1(z)^2 + z^{1/2}\tau_3(z)^2$$
$$\tau_3(qz)\tau_3(q^{-1}z) = \tau_3(z)^2 + z^{1/2}\tau_1(z)^2$$

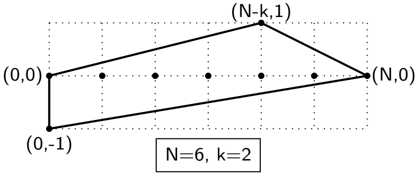
"Generic phenomenon": for the Toda family ($Y^{N,k}$-geometry)

$$\tau_j\left(qz\right)\tau_j\left(q^{-1}z\right) = \tau_j(z)^2 + z^{1/N}\tau_{j+1}\left(q^{k/N}z\right)\tau_{j-1}\left(q^{-k/N}z\right), \quad j \in \mathbb{Z}/N\mathbb{Z}$$

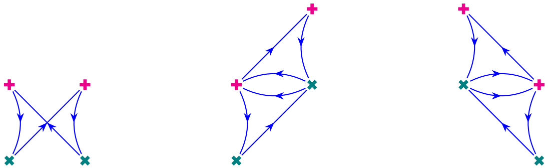
- generated by Toda discrete flows;
- are solved in terms of (dual) Nekrasov functions: "Kiev formulas" (talk by M.Bershtein).

Toda family

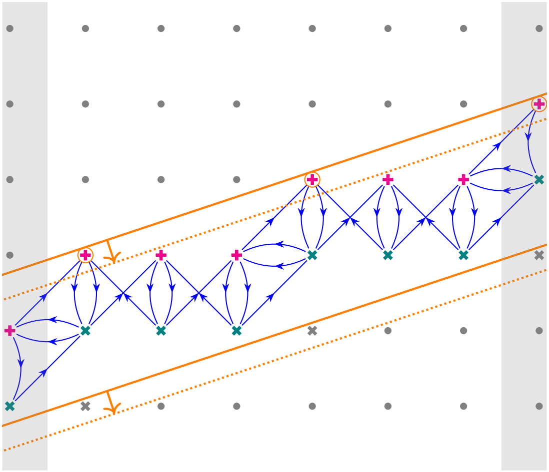
$Y^{N,k}$ polygons with $0 \le k \le N$: $B = 4$ boundary points, hyperelliptic curves.



Quivers for $Y^{N,k}$ theories can be glued from blocks of three types 0, 1, -1, respectively. $N = N_1 + N_0 + N_{-1}$, $k = N_1 - N_{-1}$.

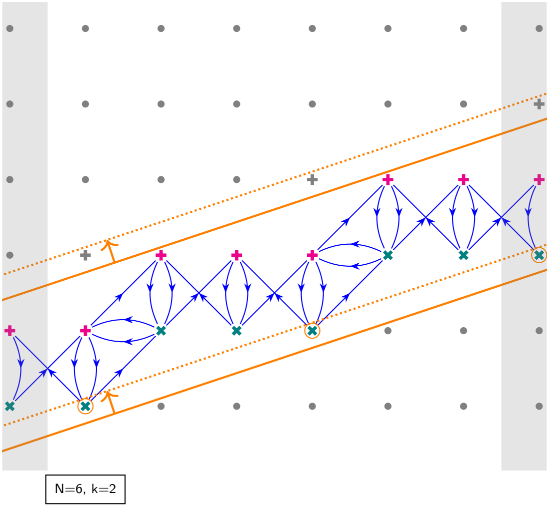


Toda discrete flow



N=6, k=2

Toda discrete flow



Differential limit

In

$$\tau_j(qz)\,\tau_j\left(q^{-1}z\right) = \tau_j(z)^2 + z^{1/N}\tau_{j+1}\left(q^{k/N}z\right)\tau_{j-1}\left(q^{-k/N}z\right)$$

take $q = \exp R$, $z = R^{2N}\mathrm{z}$ send $R \to 0$ ($5D \to 4D$):

$$(\partial_{\log \mathrm{z}})^2 \log \tau_j = \mathrm{z}^{1/N}\frac{\tau_{j+1}\tau_{j-1}}{\tau_j^2}\,, \quad j \in \mathbb{Z}/N\mathbb{Z}$$

for *any* k. From isomonodromic tau-function (talk of P.Gavrylenko!) one gets

$$\frac{d^2\phi_j}{dr^2} + \frac{1}{r}\frac{d\phi_j}{dr} = e^{\phi_{j+1}-\phi_j} - e^{\phi_j-\phi_{j-1}}$$

for $N = 2$ – radial sinh-Gordon equation (well-known form of PIII_3): for $\phi_j = \log \tau_j/\tau_{j-1}$, $\mathrm{r}=2\mathrm{N}\,\mathrm{z}^{\frac{1}{2N}}$.

Lax representation

Poisson submanifolds in $\widehat{G}/\mathrm{Ad}\widehat{H}$ (via co-extension $\widehat{G}^\sharp$): for any cyclically reduced $u = s_{j_1} \dots s_{j_l} \Lambda^k$, $s_j \in (\hat{W} \times \hat{W})^\sharp$ - the "Lax map" (Fock & AM)

$$x_1, \dots, x_l \mapsto \mathbf{E}_{j_1} \mathbf{H}_{j_1}(x_1) \cdots \mathbf{E}_{j_l} \mathbf{H}_{j_l}(x_l) \Lambda(\lambda)^k = g(\vec{x}; \lambda) T_q$$

$$\mathbf{E}_i = E_i = \exp(e_i), \qquad \mathbf{F}_i = F_i = \exp(f_i)$$

$$\mathbf{H}_i(x) = H_i(x) T_x, \quad i \neq 0$$

here $H_i(x) = x^{h^i}$, $[h^i, e_j] = [h^i, f_j] = 0$ for $i \neq j$, and

$$T_x = x^{\lambda \partial / \partial \lambda}, \qquad T_q = \prod_i T_{x_i} = q^{\lambda \partial / \partial \lambda}$$

in terms of the Chevalley generators.

Integrable situation

$q = 1$, $T_q = \mathrm{id}$ (cf. with GK, where this is due to $\partial\mathbb{T}^2 = 0$).

Lax operator $g(\boldsymbol{x};\lambda)T_q = g(\vec{x};\lambda) \in \widehat{G} \subset \widehat{G}^\sharp$ is a (λ-dependent) matrix

$$\det(g(\boldsymbol{x};\lambda) + \mu) = f_\Delta(\lambda,\mu) = 0 \qquad (10)$$

gives the spectral curve equation and generates integrals of motion.

The Poisson structure coinsides with restriction of r-matrix Poisson bracket on $\widehat{G}$ (Fock & Goncharov), and this is almost immediate proof of integrability.

q-isomonodromic deformation

Diagonalization of the Lax operator $g(\vec{x};\lambda)T_q$

- integrable case: standard spectral parameter dependent Lax matrix, enough integrals of motion *only* for $q=1$;
- how "to diagonalize" for $q\neq 1$?

The linear system ($g_\pm \in \widehat{B}_\pm \subset \widehat{G}$)

$$g(\lambda)T_q\psi(\lambda)=\psi(\lambda)\,, \quad g_+(\lambda)g_-(\lambda)\psi(q\lambda)=\psi(\lambda)\,.$$

Isomonodromic transformation $\psi(\lambda)=g_+(\lambda)\psi'(\lambda)$, then

$$g_-(\lambda)g_+(q\lambda)\psi'(q\lambda)=\psi'(\lambda)\,.$$

hence

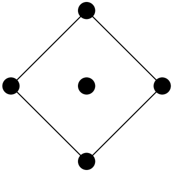
$$g'(\lambda)=g_-(\lambda)g_+(q\lambda)=g'_+(\lambda)g'_-(\lambda)\,.$$

is q-Schlesinger equation, generating discrete flow $T: x\mapsto x'$.

Summary

- q-difference equation (arising from cluster integrable systems) are more transparent, than differential ones;
- the corresponding tau-function (not of *integrable* systems!) satisfy simple Hirota equations, and ... do have solutions;
- there is q-isomonodromic system, following from Poisson structure on co-extended loop groups.

Directions of the generalization



4 boundary points, internal points on one line → Non-autonomous discrete Hirota equations

One internal point ↓ q-difference Painlevé equations

? General Newton polygons

Thank you for your attention!