An extrapolative approach to integration

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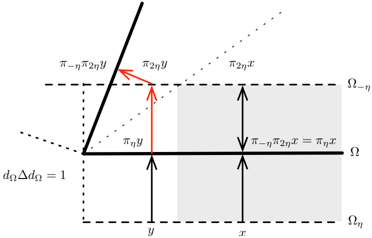
What happens at a corner?



Extrapolative approach

Recall:

$$I_0 := \int_\Gamma v(\mathbf{y}) dS(\mathbf{y}),$$

Assume

1. $\phi : \mathbb{R}^n \mapsto \mathbb{R}$, $n \in \mathbb{N}$: Lipschitz function
2. $\Gamma_\eta := \{\mathbf{x} : \phi(\mathbf{x}) = \eta\}$
3. $\tilde{v} : \mathbb{R}^n \mapsto \mathbb{R}$: Lipschitz function

Define

$$S := \int_{\mathbb{R}^n} \tilde{v}(\mathbf{x}) \delta_\epsilon(\phi(\mathbf{x})) |\nabla \phi(\mathbf{x})| d\mathbf{x}$$

$$I[\tilde{v}, \phi](\eta) := \int_{\Gamma_\eta} \tilde{v}(\mathbf{x}) dS(\mathbf{x}).$$

In general

$$S := \int_{\mathbb{R}^n} v(\mathbf{y}^*)\delta_\epsilon(d(\mathbf{y}))dS(\mathbf{y}) \neq I_0!!!!$$

Theorem (K., Tsai (2018))

Suppose

1. *d is the signed distance function to Γ*
2. *$\tilde{v}$ is constant along the normals of Γ*
3. *Γ_η are closed C^2 hypersurfaces for $-\epsilon \leq \eta \leq \epsilon$.*

Then for sufficiently small $\epsilon > 0$, we have

$$I[\tilde{v}, d](\eta) = I_0 + \sum_{i=1}^{n-1} A_i \eta^i,$$

where A_i, $1 \leq i \leq n$ are constants that depend on $\tilde{v}$ and d.

Theorem (K., Tsai (2018))

Assume the previousTheorem holds and assume δ_ϵ *is compactly supported in* $[-\epsilon, \epsilon]$ *with* $n-1$ *vanishing moments , namely*

$$\int_{-\infty}^{\infty} \delta_\epsilon(\eta)\eta^p d\eta = \begin{cases} 1 & p = 0, \\ 0 & 0 < p \leq n-1, \end{cases}$$

then

$$I_0 = \int_\Gamma v(\mathbf{x}) dS(\mathbf{x}) = \int_{\mathbb{R}^n} \tilde{v}(\mathbf{x}) \delta_\epsilon(d(\mathbf{x})) d\mathbf{x} = S.$$

Curves with corners and cusps

1. Corner: $I(\eta) = I_0 + O(\eta)$
2. Cusp: $I(\eta) = I_0 + O(\eta^{\frac{1}{p}})$ where p quantifies the degree of the cusp.

Theorem (K.,Tsai (2018))

Consider a curve Γ in $\mathbb{R}^2$ with a corner at (x_0, y_0) modeled locally by $g \in C^2([0,\infty),[0,\infty))$ with $g(0) = 0$ and for $p \in \mathbb{N}$, $g^{(\nu)}(0) = 0$ for $0 \leq \nu < p$ and $g^{(p)}(0) > 0$. Suppose also that δ_ϵ is compactly supported in $[-\epsilon, \epsilon]$ with m vanishing moments such that then for small $\epsilon > 0$

$$|S - I_0| = \begin{cases} O\left(\epsilon^{1+m}\right) & p = 1 \text{ (corner)} \\ O\left(\epsilon^{2+\frac{1}{p}}\right) & p \geq 2 \text{ (cusp)} \end{cases}.$$

Integrating a Lipschitz continuous function on a circle

Integrand:

$$f(x,y) = \min(|\theta - 0.3|, |\theta - 2\pi - 0.3|), \quad 0 \le \theta = \arg(x,y) < 2\pi.$$

with the signed distance function to the circle.
Use a C^∞ kernel with two vanishing moments.

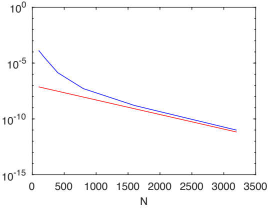


Figure: In blue: relative errors. In red: graph of $0.997^N 10^{-7}$.

Surface area of $\phi(x, y, z) := |x| + |y| + |z| = r_0$ with $r_0 = 0.65$ (ℓ_1-ball)

Use a C^∞ kernel with two vanishing moments.

Table: Relative error in computing the surface area of an ℓ_1-ball.

	N=100	200	400	800
Rel. error	5.87232e-1	2.63126e-2	8.19894e-4	5.23091e-6
Order		4.5	5.0	7.3

THANK YOU!