Free boundary problems as parabolic integro-differential equations

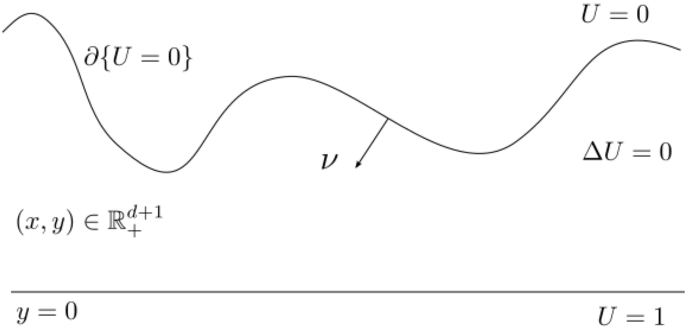
Nestor Guillen
University of Massachusetts

June 2018

What follows is based on recent joint projects with Russell Schwab, Jun Kitagawa, and Héctor Chang-Lara.

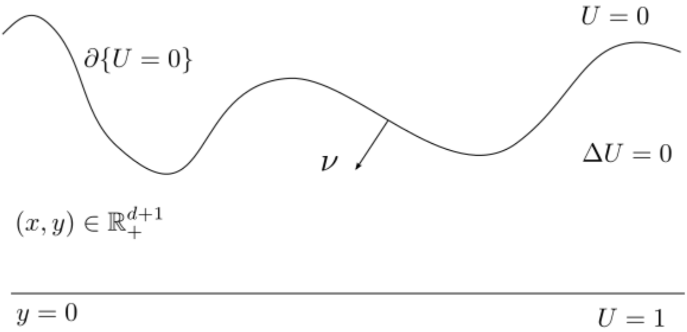
Cold opening

Consider a (one-phase) free boundary problem in $\mathbb{R}^{d+1}_+$



Cold opening

If the initial interface is given by a graph $y = f(x)$, then it is well known (by the comparison principle) that the interface will remain a graph for all later times.



Cold opening

Theorem (with Schwab and Chang-Lara, forthcoming)

The graph of $f(x,t)$ represents the interface for a solution of the free boundary problem if and only if it is a solution of the non-local (degenerate) parabolic equation

$$\partial_t f = I(f) \text{ in } \mathbb{R}^d \times [0,\infty)$$

Cold opening

Moreover, the operator $I(f)$ admits a min-max formula

$$I(f) = \min_i \max_j \{h_{ij} + L_{ij}(f)\}$$

Here:

- $h_{ij} \in \mathbb{R}$ and $\sup |h_{ij}| < \infty$.
- there are numbers $c_{ij} \leq 0$ and Lévy measures such that

$$\begin{aligned} L_{ij}(f) &= c_{ij} f(x) \\ &\quad + \int_{\mathbb{R}^d} f(x+h) - f(x) - \chi_{B_1(0)}(h) \nabla f(x) \cdot h \, d\nu_{ij}(h) \end{aligned}$$

$$\sup_{ij} \int_{\mathbb{R}^d} \min\{1, |h|^{1+\varepsilon}\} \, d\nu_{ij}(h) < \infty$$

Cold opening

Theorem

Suppose that $f(x,0)$ admits a modulus of continuity ω, then $f(x,t)$ admits the same modulus of continuity for all $t > 0$.

Note: This result exploits the fact that $c_{ij} \leq 0$ in the min-max formula.

Remarks

1. More than non-locality, this result is about the comparison principle.
2. The result paves the way to applying non-local regularity theory (concretely Krylov-Safonov type results) to analyze the interface of free boundary problems.
3. All of the above results include two-phase problems and problems with (some) nonlinearities.

Remarks

4. This approach **could** be extended to free boundaries that are not given by a graph over $\mathbb{R}^d$. The resulting parabolic equation would take place in a reference submanifold. However, due to the complicated geometry we expect this representation will only holds for short times.
5. The min-max expression is not explicit, so as matters stand, this description is of **no use** for performing numerical computations.

Background

It is worth comparing this with approaches based on the Hanzawa transform

$$\Gamma(t) = \{x + h(x,t)\nu_{\Gamma_0} \mid x \in \Gamma_0\}$$

Here Γ_0 is the reference interface, $h(x,t)$ is used to construct a diffeomorphism to set the FB in a fixed domain, yielding a coupled system involving $h(x,t)$ and the other transformed variables.
This approach does not depend on the comparison principle structure, and accordingly is able to treat problems with surface tension (e.g. work of Escher-Simonett).

Background

Another perspective that involves non-local equations arises in the Muskat problem. There, Córdoba and Gancedo showed the FB problem reduces to the non-local equation

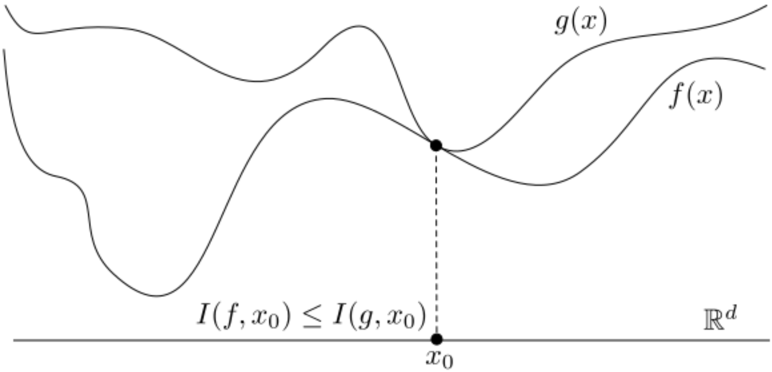
$$\partial_t f = \frac{1}{4\pi}\text{P.V.}\int_{\mathbb{R}^2} \frac{(\nabla f(x,t) - \nabla f(x-h,t))\cdot h}{(|h|^2 + (f(x,t) - f(x-h,t)^2))^{\frac{2}{3}}}\,dh,$$

which is clear linearizes to the fractional heat equation. This fact is responsible for a number of well-posedness and regularity results for the Muskat problem over the past decade.

The Global Comparison Property

A map $I : C_b^2(\mathbb{R}^d) \mapsto C_b^0(\mathbb{R}^d)$ has the **Global Comparison Property (GCP)** if:

$$g \text{ touches } f \text{ from above at } x_0 \Rightarrow I(f, x_0) \leq I(g, x_0).$$



The Global Comparison Property

Examples

- The Laplacian $I(f,x) = \Delta f(x)$.
- Any drift-diffusion operator $I(f,x) = \mathrm{tr}(a(x)D^2 f(x)) + b(x) \cdot \nabla f(x)$.
- Any Hamiltonian operator $I(f,x) = H(\nabla f(x), x)$.

The Global Comparison Property

Examples

- Any *Hamilton-Jacobi-Bellmann operator*, such as

$$I(f,x) = \min\{\Delta f(x), \Delta f(x) + 5\partial_{x_2x_2} f(x)\}$$
$$I(f,x) = \min_i \max_j \{\mathrm{tr}(a_{ij} D^2 f(x)\}$$

(there arise in stochastic control and differential games)

- Fractional powers of the Laplacian $-(\Delta)^{\frac{\alpha}{2}}$
- Any finite difference operator, e.g.

$$I(f,x) = f(x+y_0) - f(x), \;\; y_0 \in \mathbb{R}^d.$$

The Global Comparison Property

Lévy operators

A Lévy operator is a linear map $L : C_b^2 \mapsto C_b$ of the form

$$L\phi = c(x)\phi(x) + b(x) \cdot \nabla \phi(x) + \mathrm{tr}(a(x)D^2\phi(x)) \\ + \int_{\mathbb{R}^d \setminus \{0\}} \phi(x+h) - \phi(x) - \chi_{B_1(0)} \nabla u(x) \cdot h \, d\mu_x(h)$$

where $a, b, c \in L^\infty$, $a(x) \geq 0$, and μ_x denotes for every $x \in \mathbb{R}^d$ a **Lévy measure**

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} \min\{1, |h|^2\} d\mu_x(h) < \infty$$

The Global Comparison Property

Theorem (Courrège)

If $L : C_b^2(\mathbb{R}^d) \mapsto C_b^0(\mathbb{R}^d)$ is a bounded linear map having the GCP, then L is a Lévy operator.

The Global Comparison Property

Theorem (Courrège)

If $L : C_b^2(\mathbb{R}^d) \mapsto C_b^0(\mathbb{R}^d)$ is a bounded linear map having the GCP, then L is a Lévy operator.

Lévy operators with constant coefficients and $c = 0$	$\Leftrightarrow$	Generators of Lévy processes

This much is the content of the Lévy-Khintchiner formula

The Global Comparison Property

Beyond $\mathbb{R}^d$

Maps with the GCP arise naturally in contexts other than $\mathbb{R}^d$

The Global Comparison Property

Beyond $\mathbb{R}^d$

Given (say) a compact metric space (X, d), a map

$$I : Y \subset C(X) \to C(X)$$

is said to have the GCP if $u(x) \leq v(x)$ for all $x \in X$ with $u(x_0) = v(x_0)$ at some $x_0 \in X$ implies

$$I(u, x_0) \leq I(v, x_0).$$

The Global Comparison Property

A few more examples

Graphs (G, ω_{ij})	$\sum_{j \in G} \omega_{ij}(f_j - f_i)$
Fractals (e.g. Sierpinski's)	$\lim_{k \to \infty} 5^k \frac{3}{2}(-4f(x) + \sum_{y \sim_k x} f(y))$
Riemannian mfolds. (M, g)	$\Delta_g f$ (Laplace-Beltrami op) $P_\gamma f$ (Paneitz op.)
Hypersurfaces $(\Sigma \subset \mathbb{R}^d)$	$\Delta_\Sigma f$ (Laplace-Beltrami op.) $\partial_n U_f$ (Dirichlet-to-Neumann map)

The Global Comparison Property

Dirichlet to Neumann map revisited

Let $X = \partial\Omega$, for some smooth domain $\Omega \subset \mathbb{R}^d$,
$F : \mathrm{Sym}(d) \to \mathbb{R}$ a uniformly elliptic operator.

Given $\phi \in C^{1,\alpha}(\partial\Omega)$ ($\alpha > 0$), let U_ϕ be the viscosity solution to

$$\begin{cases} F(D^2 U_\phi) & = 0 \quad \text{in } \Omega \\ U_\phi & = \phi \quad \text{on } \partial\Omega. \end{cases}$$

Then, if n denotes the inner normal to $\partial\Omega$, set

$$I(\phi, x) := \partial_n U_\phi(x), \quad \forall\, x \in \partial\Omega.$$

This map $I : C^{1,\alpha}(\partial\Omega) \to C(\partial\Omega)$ is Lipschitz and has the GCP.

Min-max formula for maps with the GCP

Many of the most interesting examples in the preceding discussion were not linear– such as the Dirichlet to Neumann map. Can you still prove a similar characterization as what Courrège proved for linear operators?

If I is **local**, this has been known and used for years.

If I is not assumed to be local, it was not known.

Yes, if you assume I is Lipschitz.

Lipschitz maps with the GCP

A min-max formula

Theorem (with Schwab)

Let M be a complete, d-dimensional manifold, and let $I : C_b^2(M) \to C_b^0(M)$ be Lipschitz, with the GCP. Then

$$I(u,x) = \min_i \max_j \{h_{ij}(x) + L_{ij}(u,x)\} \;\; \forall \, u, x.$$

Moreover, for each pair of indices ij, we have (uniformly)

- $h_{ij}(x) \in C_b^0(\mathbb{R}^d)$
- $L_{ij} : C_b^2(\mathbb{R}^d) \to C_b^0(\mathbb{R})$ *is a Lévy operator*

Lipschitz maps with the GCP

A min-max formula

Theorem (with Schwab)

Furthermore if $I : C^{1,\gamma}(\mathbb{R}^d) \to C(\mathbb{R}^d)$ is Lipschitz and satisfies the GCP, then the L_{ij}'s have the form

$$\begin{aligned} L(u,x) = {} & C(x)u(x) + B(x)\cdot\nabla u \\ & + \int_{\mathbb{R}^d} u(x+y) - u(x) - \nabla u(x)\cdot y\chi_{B_1(0)}\,\nu(x,dy) \end{aligned}$$

and

$$\sup_x \int \min\{|y|^{1+\gamma},1\}\,\nu(x,dy) < \infty.$$

Lipschitz maps with the GCP

A min-max formula

Theorem (with Schwab)

Furthermore if $I : C^{0,\gamma}(\mathbb{R}^d) \to C(\mathbb{R}^d)$ *is Lipschitz and satisfies the GCP, then the* L_{ij}*'s have the form*

$$L(u,x) = C(x)u(x) + \int_{\mathbb{R}^d} u(x+y) - u(x)\ \nu(x,dy)$$

and

$$\sup_x \int \min\{|y|^\gamma, 1\}\ \nu(x,dy) < \infty.$$

The importance of min-max formulas

Viscosity solutions $\Leftrightarrow$ Value functions

For local elliptic equations

$$F(D^2u, \nabla u, u, x) = 0$$

we have that the left side can be represented as

$$\min_i \max_j \{h_{ij}(x) + c_{ij}(x)u(x) + \nabla u \cdot b_{ij}(x) + \mathrm{tr}(A_{ij}(x)D^2u(x))\}$$

In this setting, min-max formulas have been of great use, as they allow us to represent solutions to a PDE as the value functions of zero-sum differential games, i.e.

Fully nonlinear elliptic equation	$\Leftrightarrow$	Isaacs equation for some game

The importance of min-max formulas

Viscosity solutions ⇔ Value functions for games

Tools + Problems for fully nonlinear PDE	⇔	Tools + Problems in Stochastic Control & Differential Games

The importance of min-max formulas

Results relying on the (known) local min-max

- Existence nonlinear first order equations via value function in a stochastic differential game and the vanishing viscosity: Fleming 1969, Friedman 1974.

- Accretive operator method of Evans 1980.

- Hamilton-Jacobi equations, “blow-up” limits, structure of level sets, geometric motions, “generalized” characteristics, and finite domain/cone of dependence: Evans-Ishii 1984, Evans Souganidis 1984, Lions-Souganidis 1985.

The importance of min-max formulas

Results relying on the (known) local min-max

(...continued)

- Finite difference schemes: Kuo-Trudinger 2007, Krylov 2015.

- Homogenization, Lions-Papanicolaou-Varadhan 1980's.

- Existence/regularity of viscosity solutions, Katsoulakis 1995.

- Fully nonlinear second order parabolic equations and a class of deterministic two-player games, Kohn-Serfaty 2006, 2010.

The importance of min-max formulas

Results that **assume** a min-max representation.

• Uniqueness of viscosity solutions: Jakobsen-Karlsen 2006, Barles-Imbert 2010.

• Properties of viscosity solutions, Caffarelli-Silvestre, 2009.

• All Krylof–Safonov/Evans–Krylov type regularity results.

• Critical nonlocal drift diffusion, Silvestre 2011.

• Integro-differential homogenization, Schwab 2010, 2012

• Relationship between viscosity solutions and differential games of jump process, Koike-Swiech 2013

The importance of min-max formulas

Regularity theory and extremal operators

A word on the connnection with **regularity theory**: let $\mathcal{L}$ denote a family of linear operators of the form

$$L(u,x) = \int_{\mathbb{R}^d} (u(x+y) - u(x))K(y)\,dy, \;\; K \in \mathcal{K}.$$

Then, a nonlinear operator I is said to be uniformly elliptic with respect to $\mathcal{L}$ if

$$M_{\mathcal{L}}^{-}(u-v,x) \leq I(u,x) - I(v,x) \leq M_{\mathcal{L}}^{+}(u-v,x),$$

where $M_{\mathcal{L}}^{\pm}$ denote the **extremal operators** for $\mathcal{L}$:

$$M_{\mathcal{L}}^{+}(\phi,x) := \sup_{L \in \mathcal{K}} L(\phi,x),$$
$$M_{\mathcal{L}}^{-}(\phi,x) := \inf_{L \in \mathcal{K}} L(\phi,x).$$

Lipschitz maps with the GCP

The importance of min-max formulas...

...it is not hard to see that

$$M^-_{\mathcal{L}}(u-v,x) \leq I(u,x) - I(v,x) \leq M^+_{\mathcal{L}}(u-v,x)$$

is equivalent to I being expressable by a min-max

$$I(u,x) = \min_i \max_j \{h_{ij} + L_{ij}u\}$$

where every L_{ij} belongs to $\mathcal{L}$.

Characterizing those families $\mathcal{L}$ leading to a Harnack ineq. or Hölder estimates is an important unresolved question.

About the proof of the min-max

Ideas behind the proof

The min-max formula in turn reduces to the following assertion:

There is a class $\mathcal{L}$ of linear operators $C_b^2(\mathbb{R}^d) \mapsto C_b^0(\mathbb{R}^d)$ with the GCP, such that if $u, v \in C_b^2(\mathbb{R}^d), x \in \mathbb{R}^d$, there is $L \in \mathcal{L}$ with

$$I(u,x) - I(v,x) \leq L(u-v,x).$$

In this case, it is immediate that

$$I(u,x) = \min_{v \in C_b^2(\mathbb{R}^d)} \max_{L \in \mathcal{L}} \{I(v,x) + L(u-v,x)\}$$

About the proof of the min-max

Ideas behind the proof

...In this case, it is immediate that

$$I(u,x) = \min_{v \in C_b^2(\mathbb{R}^d)} \max_{L \in \mathcal{L}} \{I(v,x) + L(u-v,x)\}.$$

Then, the min-max formula would hold with index sets for a and b given by $C_b^2(\mathbb{R}^d)$ and $\mathcal{L}$, respectively, with

$$f_{vL} := I(v,x) - L(v,x), \quad L_{vL} := L.$$

About the proof of the min-max

Ideas behind the proof

The existence of such a family follows easily **when I is Fréchet differentiable**, noting that

1) the Fréchet derivative of I at any u_0 inherits the GCP:

$$\begin{aligned}
&u \text{ touches } v \text{ from above at } x_0\\
&\Rightarrow u_0 + tuu \text{ touches } u_0 + tv \text{ from above at } x_0\\
&\Rightarrow I(u_0 + tu, x_0) \leq I(u_0 + tv, x_0) \;\; \forall\, t > 0\\
&\Rightarrow \frac{d}{dt\,|_{t=0}} I(u_0 + tu, x_0) \leq \frac{d}{dt\,|_{t=0}} I(u_0 + tv, x_0)
\end{aligned}$$

About the proof of the min-max

Proof for smooth I

The existence of such a family follows easily **when I is Fréchet differentiable**, noting that

2) we may differentiate+integrate, obtaining the identity

$$\begin{aligned} I(u,x) - I(v,x) &= \int_0^1 \frac{d}{dt}\left(I(v+t(u-v),x)\right)\, dt \\ &= \left(\int_0^1 DI(v+t(u-v))\, dt\right)(u-v,x), \end{aligned}$$

where $L = \int_0^1 DI(v+t(u-v))dt$ is bounded and has the GCP.

About the proof of the min-max

Proof for smooth I

Taking

$$\mathcal{L} = \text{hull}\{L : C_b^2(\mathbb{R}^d) \mapsto C_b^0(\mathbb{R}^d) \mid L = DI(u), \;\; u \in C_b^2(\mathbb{R}^d)\},$$

we have

$$I(u,x) - I(v,x) \leq \max_{L \in \mathcal{L}} L(u-v,x),$$

as we wanted.

About the proof of the min-max

Proof for Lipschitz I

When I is merely Lipschitz things are not so simple.

The chief reason (but not the only one):

Lipschitz maps between infinite dimensional Banach spaces may not be Fréchet differentiable in ***any*** *dense set.*

About the proof of the min-max

Proof for Lipschitz I

Lipschitz maps between infinite dimensional Banach spaces may not be Fréchet differentiable in ***any*** *dense set.*

Most of the theorem's proof consisted in working around this!

About the proof of the min-max

Proof for Lipschitz I

Lipschitz maps between infinite dimensional Banach spaces may not be Fréchet differentiable in ***any*** *dense set.*

Most of the theorem's proof consisted in working around this!

Outline:

- Prove a "finite dimensional" analogue for (finite) graphs.
- Approximate $\mathbb{R}^d$ or M by a certain sequence of graphs.
- Approximate C_b^2 via the space of functions on the graphs and the map I by finite dimensional Lipschitz maps –in a way that approximately preserves ordering and the GCP!!.
- Pass to the limit and "lift" the finite dim. min-max to $C^2(\mathbb{R}^d)$

Dirichlet to Neumann Maps

Consider Ω, a domain with smooth boundary

Given $f \in C^{1,\alpha}(\partial\Omega)$ there is a unique viscosity sol. U_f of

$$\begin{cases} F(D^2U_f, DU_f, U_f, x) & = 0 \text{ in } \Omega \\ U_f & = f \text{ on } \partial\Omega \end{cases}$$

Define the Dirichlet-to-Neumann Map for F, via

$$I(f, x) = \partial_\nu U_f(x)$$

Where ∂_ν denotes the (inner) normal derivative on $\partial\Omega$.

Dirichlet to Neumann Maps

In this case, the min-max formula says that

$$I(f,x) = \min\max\{h_{ij}(x) + L_{ij}(f,x)\},$$

for continuous functions h_{ij} and Lévy operators L_{ij}

$$L_{ij}(f,x) = C_{ij}(x)f(x) + (B_{ij}(x), \nabla f(x)) \\ + \int_{\partial\Omega} u(y) - u(x) - (\nabla u(x), \exp_x(y))\chi_{B_r(x)}(y)\, \mu_{ij}(x,dy)$$

Dirichlet to Neumann Maps

Question: What can be said about the measures $\mu_{ij}(x, dy)$? Are they absolutely continuous with respect to surface measure?

This turns out to be a quite difficult question, why? well:

measures $\mu_{ij}(x, dy) \approx$ normal der. L-harm. meas. of some L

For a nonlinear equation, the respective L could have rough coefficients: possibly very singular L-harmonic measures!

Dirichlet to Neumann Maps

Consider the special case where F is a linear, non-divergence form operator, that is, U_f solves

$$\mathrm{tr}(A(x)D^2U(x)) + B(x)\cdot DU(x) + C(x)U(x) - D(x) = 0 \text{ in } \Omega.$$

Assume: A, B, are Hölder continuous, $A(x) \geq \lambda \mathrm{I}$, and C, D are bounded.

Dirichlet to Neumann Maps

Theorem (with Kitagawa and Schwab, 2017)

In the linear case described above, we have

$$\mu(x, dy) = k(x, y)\, d\sigma(y)$$

Moreover, there are positive constants c, C *such that*

$$c|x-y|^{-d-1} \leq k(x,y) \leq C|x-y|^{-d-1}$$

Provided $x, y \in \partial\Omega$ *and* $|x-y| \leq 1$.
The constants depend only on the dimension, Ω*, and the bounds on the coefficients.*

Dirichlet to Neumann Maps

A consequence of this result is that

Known (pointwise, regularity...) estimates for integro-differential problems	$\Rightarrow$	**New** (pointwise, regularity...) estimates **at the boundary** for Neumann problems

Dirichlet to Neumann Maps

Example: Let

$$Lu = \operatorname{tr}(A(x)D^2u(x)) + B(x) \cdot Du(x) + C(x)u(x)$$

Let $u : \Omega \times [0, T] \mapsto \mathbb{R}$ be a viscosity solution of

$$\begin{aligned} Lu &= 0 \text{ in } \Omega \\ \partial_t u &= G(\partial_n u, x, t) \text{ on } \partial\Omega \end{aligned}$$

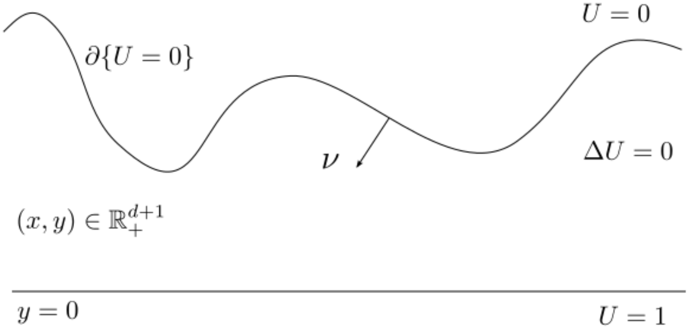
Then, u is Hölder continuous in space and time, with

$$[u]_{C^\alpha(\partial\Omega \times [T/2,T])} \leq C\left(\|u\|_{L^\infty} + \|G(0, x, t)\|_{L^\infty}\right)$$

Free boundary problems

FBs as a parabolic integro-differential equation

Let us go back to free boundary problems in $\mathbb{R}^{d+1}_+$, for simplicity, we consider the one-phase Hele-Shaw.



Then, $U : \mathbb{R}^{d+1}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a non-negative function solving

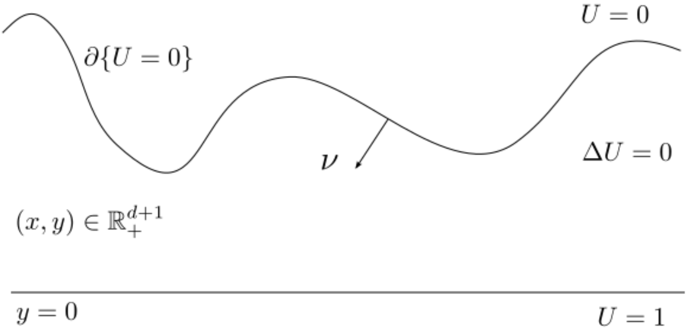
$$\text{(HS)} \begin{cases} \Delta U &= 0 \qquad \text{in } \{U > 0\}, \\ U &= 1 \qquad \text{on } \{y = 0\}, \\ V &= |\nabla U| \quad \text{on } \partial\{U > 0\}. \end{cases}$$

V denoting the normal velocity of the free boundary $\partial\{U > 0\}$.

Free boundary problems

FBs as non-local parabolic equations

Recall that if $\partial\{U_0 > 0\}$ is given by a graph in x, then the same is true of $\partial\{U(\cdot, t)\}$ for all $t > 0$.



Free boundary problems

FBs as non-local parabolic equations

Let $f(x,t)$ ($(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$) be such that

$$\{U > 0\} = \{(x,y) \mid 0 < y < f(x,t)\},$$

and let's see f solves a non-local parabolic equation.

Free boundary problems

FBs as non-local parabolic equations

The equation for f resembles the Dirichlet to Neumann map!

Given $f : \mathbb{R}^d \to \mathbb{R}$, continuous and positive, define the sets

$$\Omega(f) := \{(x, y) \in \mathbb{R}^{d+1} \mid 0 < y < f(x)\}$$
$$\Gamma(f) := \{(x, y) \in \mathbb{R}^{d+1} \mid y = f(x)\}.$$

Free boundary problems

FBs as non-local parabolic equations

From f to $\Omega(f)$, and to U_f: first, take U_f as the unique solution to the Dirichlet problem

$$\begin{cases} \Delta U_f &= 0 \text{ in } \Omega(f), \\ U_f &= 1 \text{ on } \{y = 0\}, \\ U_f &= 0 \text{ on } \Gamma(f), \end{cases}$$

and extend it to be identically zero in $\mathbb{R}^{d+1} \setminus \Omega(f)$.

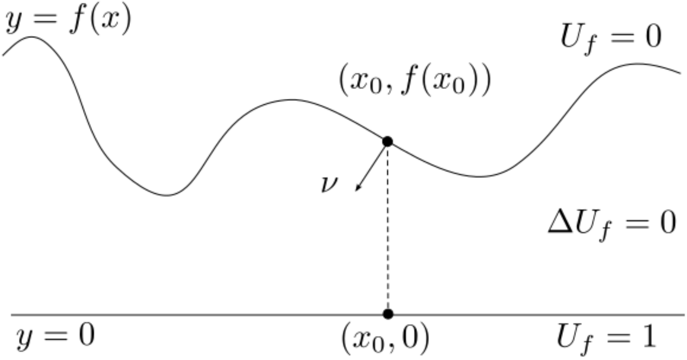
Then, define a new function on $\mathbb{R}^d$, denoted by $I(f, x)$, by

$$I(f, x) := |\nabla U_f(x, f(x))|,$$

the gradient computed from inside $\Omega(f)$ only.

Free boundary problems

FBs as non-local parabolic equations



Then (computing ∇U_f from the positivity set) define $I(f, x)$, by

$$I(f, x) := |\nabla U_f(x, f(x))|.$$

It is not difficult to show the following:

Proposition

If $f(x,t)$ is sufficiently smooth and solves

$$\partial_t f(x,t) = \frac{I(f(\cdot,t),x)}{\sqrt{1+|\nabla f(x,t)|^2}} \text{ in } \mathbb{R}^d \times \mathbb{R}_+$$

then $U(x,t) := U_{f(\cdot,t)}(x)$ will solve the Hele-Shaw problem (HS).

Free boundary problems

FBs as non-local parabolic equations

It is not difficult to show the following:

Proposition

If $f(x,t)$ is sufficiently smooth and solves

$$\partial_t f(x,t) = \frac{I(f(\cdot,t),x)}{\sqrt{1+|\nabla f(x,t)|^2}} \text{ in } \mathbb{R}^d \times \mathbb{R}_+$$

then $U(x,t) := U_{f(\cdot,t)}(x)$ will solve the Hele-Shaw problem (HS).

So, through the operator I, one can recast the Hele-Shaw problem solely in terms of the free boundary, understood here as the graph of the function f.

Hele-Shaw as an integro-differential equation

Theorem (with Schwab and Chang-Lara)

A FB problem (HS) is equivalent to the evolution equation

$$\partial_t f = If,$$

for an operator I that, for $\phi \in C^{1,\alpha}(\mathbb{R}^d)$, is given by

$$I\phi(x) := \min_i \max_j \{h_{ij} + L_{ij}(\phi, x)\}$$

Here $h_{ij} \in \mathbb{R}$, and $\{L_{ij}\}_{ij}$ are Lévy operators.

Hele-Shaw as an integro-differential equation

Regularity of the free boundary

Regularity for fully nonlinear degenerate integro-differential equations	$\Rightarrow$	Free boundary regularity for problems like one phase Hele-Shaw

This comes down to regularity estimates for solutions of

$$\partial_t f = I(f), \quad f : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}.$$

where $I : C_b^{1,\alpha}(\mathbb{R}^d) \mapsto C_b^0(\mathbb{R}^d)$ is as before.

Hele-Shaw as an integro-differential equation

Applying the Integro-Differential Theory

To apply the theory to FBs, we would need to either:

1) Show that the Lévy operators arising in the min-max formula for I (the free boundary operator) all lie within a class handled by the known regularity theorems.

2) If the above is not possible, prove as much as possible about the class of Lévy operators, and try extending the regularity theory to cover such a class (this seems very much out of reach at the moment).

What's next?

- We need methods to obtain bounds on the Lévy measures μ_x^{ij} for a generic I.
- Specific important examples: Dirichlet to Neumann map, operators arising from free boundary problems.
- The examples underline the necessity for a regularity theory for integro-differential equations on manifolds.
- Also worth considering: operators with the GCP in metric spaces (this would encompass Δ on fractals)
- What about operators with a **spatio-temporal GCP**? A min-max formula in such a setting would allow us to treat the Stefan problem as a nonlinear nonlocal space-time equation, in analogy with arguments for Hele-Shaw.

Thank You!