Activated Dynamics of the Cascading 2-GREM Scaling limits & Aging

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Spin Glasses and Related Topics,

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2-level GREM Hamiltonian

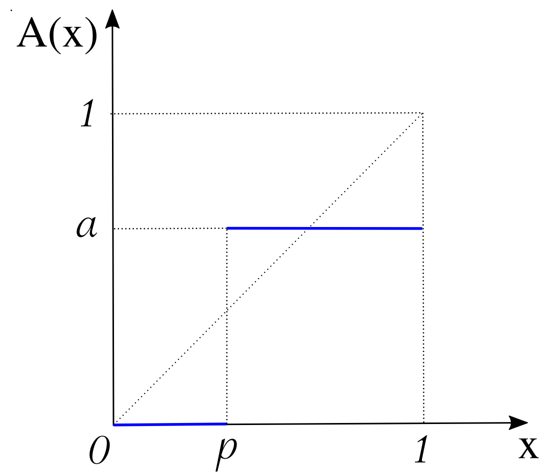
- $\mathcal{V}_n = \{-1, 1\}^n$.
- $(H_n(\sigma), \sigma \in \mathcal{V}_n)$ centered gaussian process on $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{E}\left(H_n(\sigma)H_n(\eta)\right) = nA(1 - \rho_n(\sigma, \eta)),$$

where

$$\rho_n(\sigma, \eta) = 1 - n^{-1}\left(\min\{i \mid \sigma(i) \neq \eta(i)\} - 1\right)$$

and



Alternatively, setting $n_1 = \lfloor pn \rfloor$, $n = n_1 + n_2$, $\sigma = \sigma_1\sigma_2 \in \mathcal{V}_{n_1} \times \mathcal{V}_{n_2}$,

$$H_n(\sigma) = H_n^{(1)}(\sigma_1) + H_n^{(2)}(\sigma_1\sigma_2) = \sqrt{an}\, g^{(1)}_{\sigma_1} + \sqrt{(1-a)n}\, g^{(2)}_{\sigma_1\sigma_2},$$

where $\{g^{(1)}_{\sigma_1}, g^{(2)}_{\sigma_1\sigma_2}; \sigma \in \mathcal{V}_n\}$ are i.i.d. $\mathcal{N}(0, 1)$.

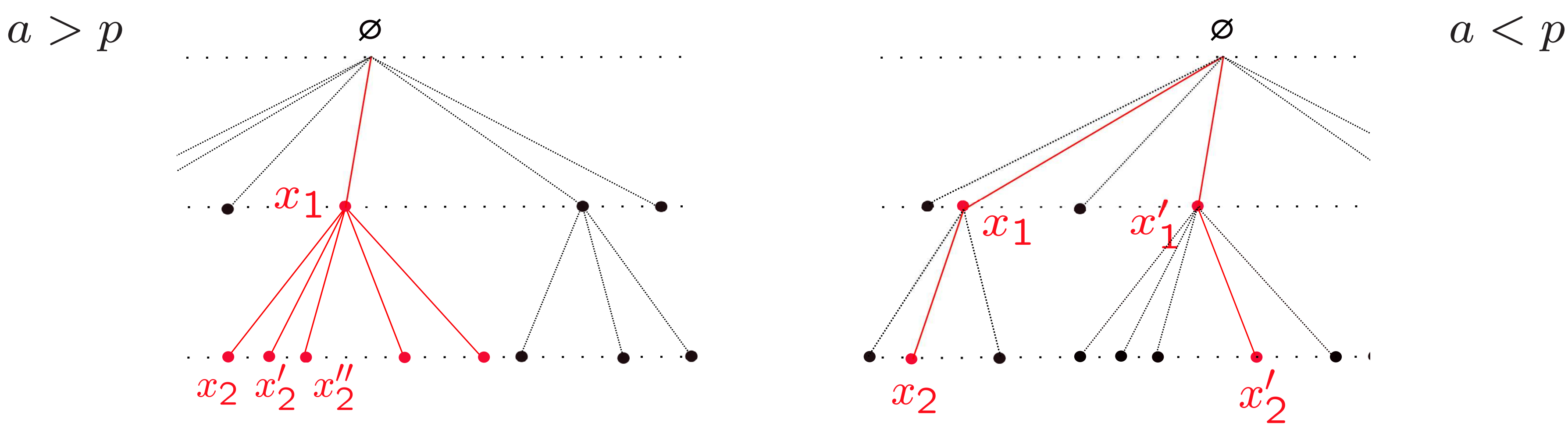
STATIC

- **Ground states** (extremes of H_n):

$$H_n^{(1)}(\sigma_1^{1}) \geq H_n^{(1)}(\sigma_1^{2}) \geq \cdots \geq H_n^{(1)}(\sigma_1^{x_1}) \geq \ldots$$

and for each $x_1 \in \{1, \ldots, 2^{n_1}\}$

$$H_n^{(2)}(\sigma_1^{x_1}\sigma_2^{x_1 1}) \geq H_n^{(2)}(\sigma_1^{x_1}\sigma_2^{x_1 2}) \geq \cdots \geq H_n^{(2)}(\sigma_1^{x_1}\sigma_2^{x_1 x_2}) \geq \ldots$$



- **Free energy for $a > p$** exhibits two discontinuities at the critical inverse temperatures

$$\beta_1 = \beta_\star \sqrt{\frac{p}{a}} \quad < \quad \beta_\star = \sqrt{2 \ln 2} \quad < \quad \beta_2 = \beta_\star \sqrt{\frac{1-p}{1-a}}$$

Which dynamics to choose?

Glauber dynamics: Markov jump process $(X_n(t), t > 0)$ on $\mathcal{V}_n$ with rates

$$e^{\beta H_n(\sigma)}\lambda_n(\sigma,\eta) = e^{\beta H_n(\eta)}\lambda_n(\eta,\sigma) \qquad \sigma \sim \eta,$$

and $\lambda_n(\sigma,\eta) = 0$ else.

- Random Hopping:

$$\lambda_n^{\text{RH}}(\sigma,\eta) \sim e^{-\beta H_n(\sigma)}, \qquad \sigma \sim \eta.$$

- Metropolis:

$$\lambda_n^{\text{Metro}}(\sigma,\eta) \sim e^{-\beta[H_n(\eta)-H_n(\sigma)]_+}, \qquad \sigma \sim \eta.$$

Because H_n is hierarchical, i.e.

$$H_n(\sigma) = \ H_n^{(1)}(\sigma_1) + H_n^{(2)}(\sigma_1\sigma_2), \quad \sigma = \sigma_1\sigma_2 \in \mathcal{V}_{n_1} \times \mathcal{V}_{n_2},$$

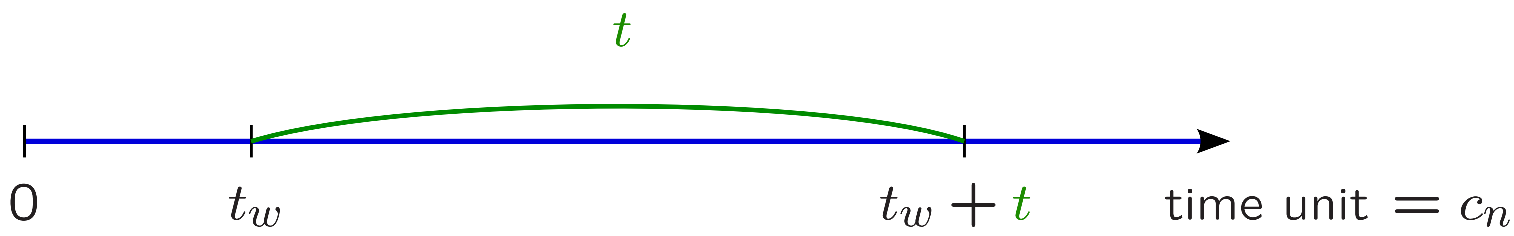
so is Metropolis

$$\lambda_n^{\text{Metro}}(\sigma,\eta) \sim e^{-\beta[H_n(\eta)-H_n(\sigma)]_+}\mathbb{1}_{\sigma \overset{1}{\sim} \eta} + e^{-\beta[H_n^{(2)}(\eta)-H_n^{(2)}(\sigma)]_+}\mathbb{1}_{\sigma \overset{2}{\sim} \eta}.$$

▶ **We choose**

$$\lambda_n(\sigma,\eta) \sim e^{-\beta H_n(\sigma)}\mathbb{1}_{\sigma \overset{1}{\sim} \eta} + e^{-\beta H_n^{(2)}(\sigma)}\mathbb{1}_{\sigma \overset{2}{\sim} \eta}.$$

AGING...

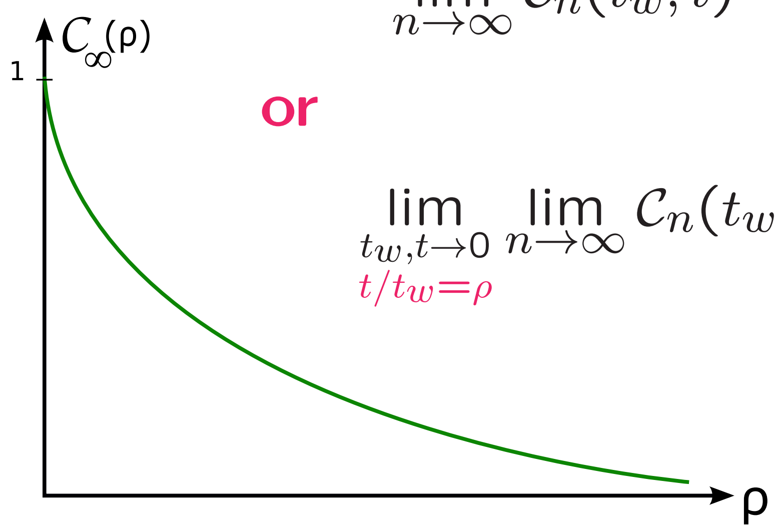


- μ_n: initial distribution
- c_n: time scale
- $\mathcal{C}_n(t_w, t)$, $t_w, t > 0$: time-time correlation function, e.g. ,

$$\mathcal{C}_n(t_w, t) = \mathcal{P}_{\mu_n}\left[n^{-1}\Big(X_n(c_n t_w), X_n(c_n(t_w + t))\Big) \geq 1 - \delta\right], \quad 0 < \delta < 1$$

...occurs if

$$\lim_{n\to\infty} \mathcal{C}_n(t_w, t) = \mathcal{C}_\infty(t/t_w),$$



or

$$\lim_{\substack{t_w, t \to 0 \\ t/t_w = \rho}} \lim_{n\to\infty} \mathcal{C}_n(t_w, t) = \mathcal{C}_\infty(\rho).$$

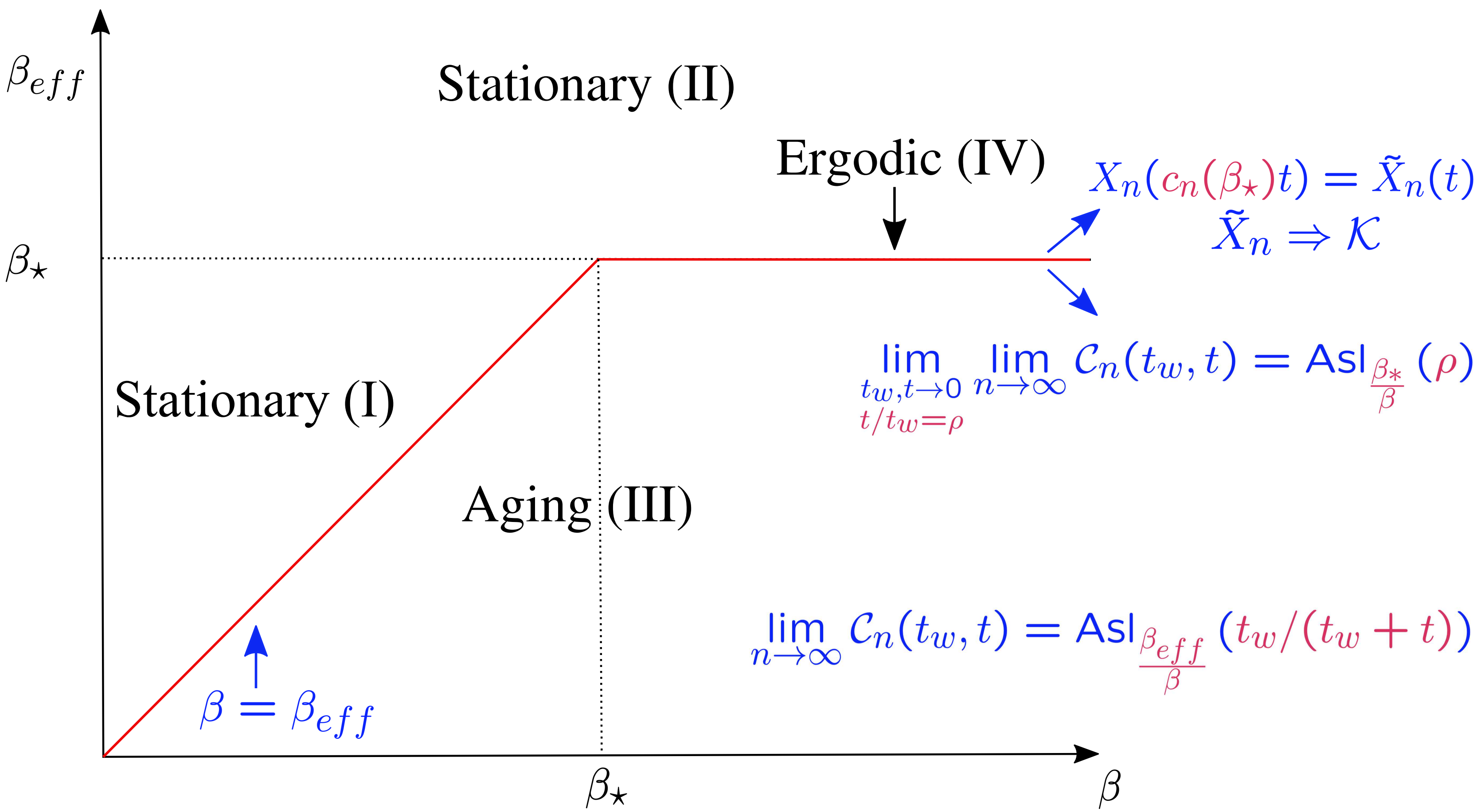
Asymptotically, $\mathcal{C}_n(t_w, t)$ is not time translation invariant

Random hopping dynamics of the REM

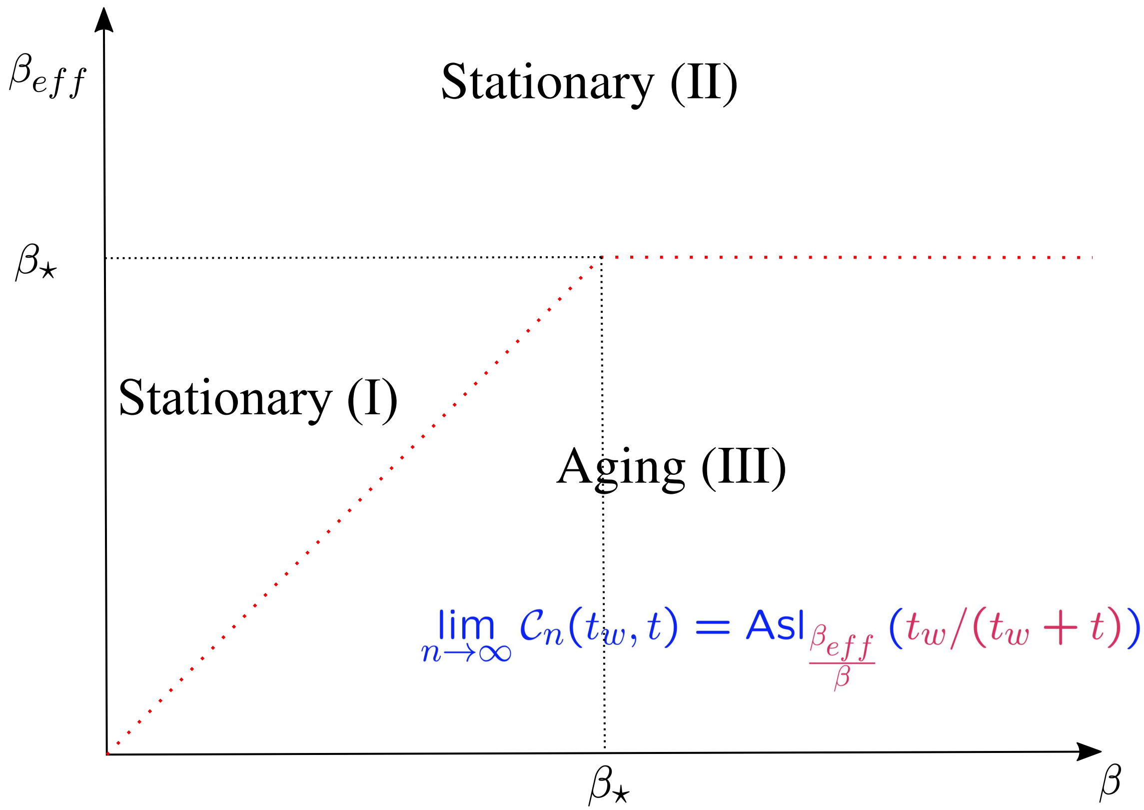
- static critical temperature: $\beta_\star = \sqrt{2\log 2}$.
- time-scale: given $\beta_{\text{eff}} > 0$,

$$c_n(\beta_{\text{eff}}) = \exp\left\{ n\beta\beta_{\text{eff}} - \frac{\beta}{2\beta_{\text{eff}}}\left(\log\left(\beta_{\text{eff}}^2 n/2\right) + \log 4\pi\right)\right\}.$$

- start $=$ uniform distribution on $\mathcal{V}_n$ if $\beta_{\text{eff}} \le \beta_\star$, any distribution else.



Metropolis dynamics of the REM



Phase transition lines are not understood: aging is proved for

$$\beta > \beta_{\text{eff}}, \quad \beta_{\text{eff}} < \beta_\star \left(1 - cst.\sqrt{\frac{\log n}{n}}\right),$$

that is, on time scales

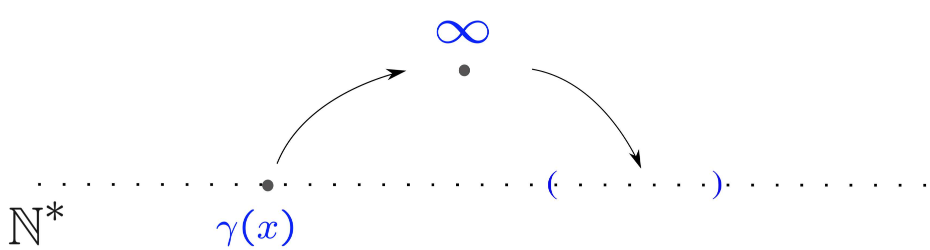
$$c_n(\beta_{\text{eff}}) < e^{-cst'\sqrt{n \log n}} c_n(\beta_\star).$$

(V.G., PTRF '18)

$\mathcal{K}$-processes

Markov processes in random environment on countably infinite state space, $\mathbb{N}^* \cup \{\infty\}$, with a single unstable state $\{\infty\}$.

• Simplest example: the uniform $\mathcal{K}$-process (as in the RHD of the REM).



• when in a stable state x, waits an exponential time of mean value $\gamma(x)$,

$$\sum_{x\in\mathbb{N}^*} \gamma(x) < \infty$$

• then jumps to the unstable state ∞, & from there, enters any finite set of stable states with uniform distribution.

• More general re-entrance mechanisms (not uniform but weighted) or state spaces (hierarchical) needed to describe more involved models or dynamics.

Ergodic (infinite volume) dynamics of mean-field spin glasses are modeled in physics through so-called (finite volume) trap models. Two GREM-like trap models were proposed (Bouchaud & Dean, Sasaki & Nemoto). Is one of them "the right one"?

Back to the cascading 2-GREM ($a > p$)

Set
$$\lambda_n(\sigma,\eta) \sim e^{-\beta(H_n^{(1)}(\sigma_1)+H_n^{(2)}(\sigma_1\sigma_2))}\mathbb{1}_{\sigma \overset{1}{\sim} \eta} + e^{-\beta H_n^{(2)}(\sigma)}\mathbb{1}_{\sigma \overset{2}{\sim} \eta}$$

$$c_n^i \sim \max_{\sigma_i \in \mathcal{V}_{n_i}} e^{\beta H_n^{(i)}(\sigma)} \approx \begin{cases} e^{n\beta\beta_\star\sqrt{ap}-\dots}, & i=1, \\ e^{n\beta\beta_\star\sqrt{(1-a)(1-p)}-\dots}, & i=2. \end{cases}$$

Two sources of dynamical phase transitions on extreme scales

- **Static:** $\beta_1 = \beta_\star\sqrt{\frac{p}{a}}$ $\quad$ $\beta_\star = \sqrt{2\ln 2}$ $\quad$ $\beta_2 = \beta_\star\sqrt{\frac{1-p}{1-a}}$

- **Fine tuning:** Given $\zeta_n < b_n \equiv \frac{n_2\beta_\star^2}{2}$ let $\tilde{\beta}_n(a,p,\zeta_n)$ be the solution in β of

$$2^{n_2}/c_n^1 = e^{\zeta_n}.$$

In explicit form:

$$\tilde{\beta}_n(a,p,\zeta_n) \sim \beta^{\mathrm{FT}}\left(1-\frac{\zeta_n}{b_n}\right), \quad \beta^{\mathrm{FT}} \equiv \frac{\beta_\star}{2}\frac{1-p}{\sqrt{pa}}.$$

We say that

$$\beta^{-1} = \tilde{\beta}_n^{-1}(a,p,\zeta_n) \text{ is } \begin{cases} \text{above} \\ \text{at} \\ \text{below} \end{cases} \text{fine tuning temperature if } \begin{cases} \zeta_n \uparrow +\infty \\ \zeta_n \sim \zeta \\ \zeta_n \downarrow -\infty \end{cases}.$$

Assume that $a > p, \beta > \beta_2$ and $\beta^{FT} > \beta_2$.

Theorem (Scaling limits). *The rescaled process* $X_n(c_n t)$ *has three possible scaling limits,* $Y = Y_1 Y_2$*, above* ($c_n = c_n^2$)*, below* ($c_n = 2^{-n_2} c_n^1 c_n^2$) *and at fine tuning* ($c_n = e^{\zeta} c_n^2$)*, all of them ergodic processes expressed through* $\mathcal{K}$*-processes. Convergence is in law w.r.t. the environment.*

Choose as time-time correlation function

$$\mathcal{C}(t_w, t) = 1\mathcal{P}(\mathcal{A}_1 \cap \mathcal{A}_2) + p\mathcal{P}(\mathcal{A}_1 \cap \mathcal{A}_2^c) + (1-p)\mathcal{P}(\mathcal{A}_1^c \cap \mathcal{A}_2)$$

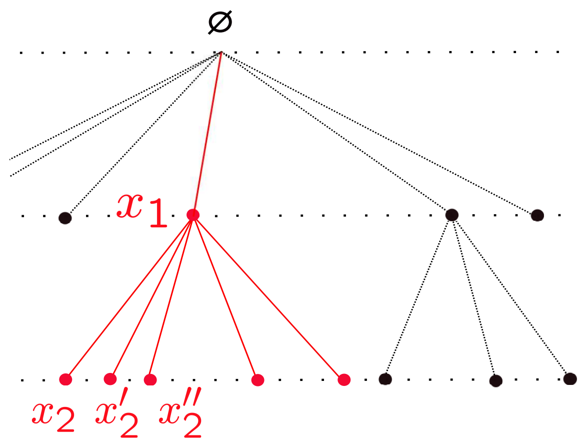
where for $i = 1, 2$,

$$\mathcal{A}_i = \mathcal{A}_i(t_w, t) = \{Y_i \text{ does not jump between times } t_w \text{ and } t_w + t\}.$$

Theorem (Aging).

$$\lim_{\substack{t_w, t \to 0 \\ t/t_w \to \rho}} \mathcal{C}(t_w, t) = \begin{cases} \mathrm{Asl}_{\alpha_2}\left(\frac{1}{1+\rho}\right), & \textit{above FT}, \\ p\,\mathrm{Asl}_{\alpha_1 \alpha_2}\left(\frac{1}{1+\rho}\right) + (1-p)\,\mathrm{Asl}_{\alpha_2}\left(\frac{1}{1+\rho}\right), & \textit{at FT}, \\ p\,\mathrm{Asl}_{\alpha_1}\left(\frac{1}{1+\rho}\right), & \textit{below FT}, \end{cases}$$

where $\alpha_i = \beta_i / \beta \in (0, 1)$ *and* $\mathrm{Asl}_\cdot$ *is the p.d.f. of the arcsine law.*



$$\gamma_n^1(x_1) = e^{\beta H_n^{(1)}(\sigma_1^{x_1})}/c_n^1$$

$$\forall x_1,\ \gamma_n^2(x_1x_2) = e^{\beta H_n^{(2)}(\sigma_1^{x_1}\sigma_2^{x_2})}/c_n^2$$

Set $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$, $\mathcal{S}_i = \{1, \ldots, M_i\}$, $i = 1, 2$. For $x = x_1x_2, y = y_1y_2 \in \mathcal{S}$,

$$\text{Proba}(x \to y) = (1 + o(1)) \begin{cases} \left[(1 - \lambda_n^{x_1}) + \lambda_n^{x_1}\,\nu_n^1(x_1)\right] \frac{1}{M_2}, & \text{if } x_1 = y_1, \\ \lambda_n^{x_1}\nu_n^1(y_1)\frac{1}{M_2}, & \text{else,} \end{cases}$$

where

$$\lambda_n^{x_1} \longrightarrow \lambda^{x_1} = \begin{cases} 1, & \text{above FT,} \\ \frac{1}{1+e^{-\zeta}M_2\gamma^1(x_1)}, & \text{at FT,} \\ 0, & \text{below FT,} \end{cases}$$

and ν_n^1 is the random measure on $\mathcal{S}_1$

$$\nu_n^1(x_1) \longrightarrow \begin{cases} \frac{\gamma^1(x_1)}{\sum_{z\in\mathcal{S}_1}\gamma^1(z)}, & \text{above FT,} \\ \frac{\gamma^1(x_1)\lambda^{x_1}}{\sum_{z\in\mathcal{S}_1}\gamma^1(z)\lambda^z}, & \text{at FT,} \\ \frac{1}{M_1}, & \text{below FT.} \end{cases}$$

None of the GREM-like trap models of theoretical physics correctly predicted the dynamics of the GREM at extreme (i.e. ergodic) time-scales.

THANK YOU FOR YOUR ATTENTION!