Critical percolation on networks with given degrees

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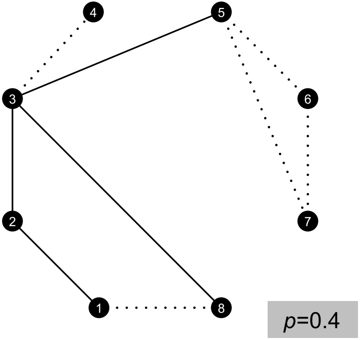
October 2, 2018

Percolation as a dynamic process

Percolation: Keep each edge in the graph with probability p, independently



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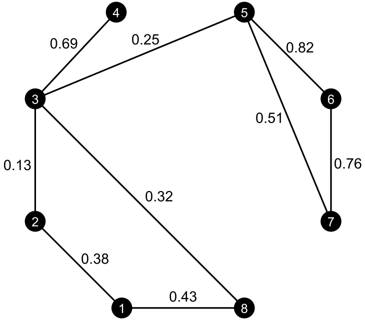
As a dynamic process (Harris coupling):

- Associate independent uniform [0,1] weights U_e to each edge e
- p as time: Keep edge e, $U_e \leqslant p$ at time p, and then increase p



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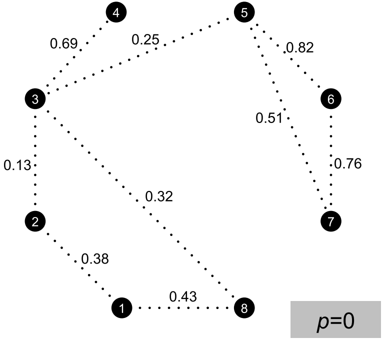

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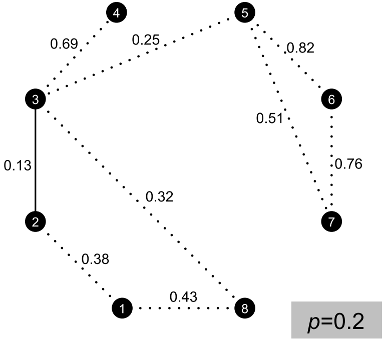

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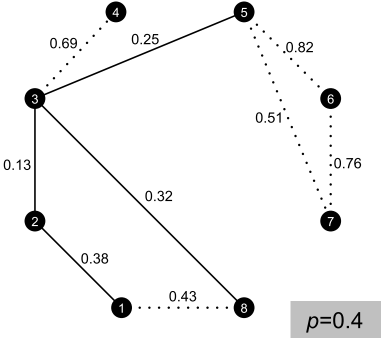

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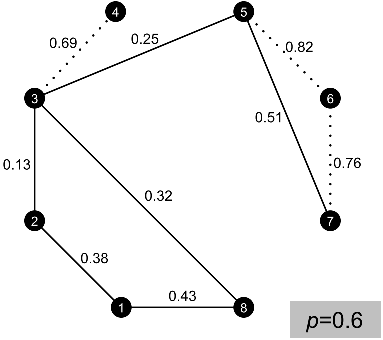

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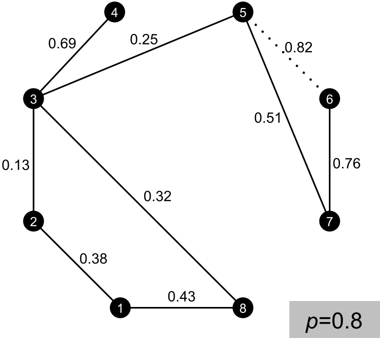

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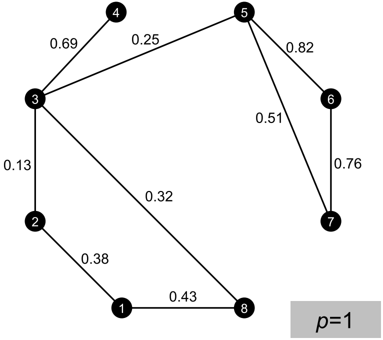

Percolation phase transition on finite graphs

There exists p_c such that for any $\varepsilon > 0$ $\qquad$ $n :=$ # vertices in the graph

(1) $p < p_c(1-\varepsilon)$: largest component is $o(n)$ — subcritical

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- ▷ Molloy & Reed (1995), Janson (2009) – Uniformly chosen graph given degree
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(1.5) $p = p_c(1 \mp \varepsilon_n)$ with $\varepsilon_n \to 0$: Critical behavior is observed

Critical window: Zoom into the critical value

Surplus edges:$=$ # edges to be deleted to turn a graph into tree

	Subcritical	Critical window	Supercritical
	$p = p_c(1 - \varepsilon)$	$p = p_c(1 \mp n^{-\eta})$	$p = p_c(1 + \varepsilon)$
Surplus edges			
Component sizes			

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Component sizes	Concentrates		Concentrates

$\varepsilon > 0$	$\varepsilon > 0$
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$\boxed{\text{Critical window: } p = p_c(1+\lambda n^{-\eta}), \quad -\infty < \lambda < \infty}$

Key questions for percolation over the critical window

$$\boxed{p = p_c(1 + \lambda n^{-\eta}), \quad -\infty < \lambda < \infty}$$

Three fundamental questions:

(Q1) Component size and surplus (at each fixed λ)

(Q2) Evolution of component size and surplus over the critical window (as λ varies)

(Q3) Graph distances within components (at each fixed λ)

Each question poses novel theoretical challenges

(Q1a) **Component size:** [Aldous & Limic '98], [Aldous & Pittel '00], [Nachmias & Peres '10], [van der Hofstad, Janssen & van Leeuwaarden '10], [Bhamidi, van der Hofstad & van Leeuwaarden '10 '12], [van der Hofstad, '13], [Bhamidi, Budhiraja & Wang '14], [Bhamidi, Sen & Wang '14], [Dembo, Levit & Vadlamani '14], [Joseph '14], [van der Hofstad & Nachmias '17] + many more...

(Q1) **Component size and surplus:** [Aldous '97], [Riordan '12], [Bhamidi, Budhiraja & Wang '14] + many more...

(Q2) **Evolution over the critical window:** [Aldous '97], [Aldous & Limic '98], [Bhamidi, van der Hofstad & van Leeuwaarden '12], [Bhamidi, Budhiraja & Wang '14], [Broutin & Marckert '16] + many more...

(Q3) **Graph distances within components:** [Nachmias & Peres 10], [Addario Berry, Broutin & Goldschmidt '12], [Bhamidi, Sen & Wang '14], [Bhamidi, van der Hofstad & Sen '17], [Broutin, Duquesne, & Wang '18] + many more...

New Challenges: Inhomogeneity in degree distribution

Inhomogeneity or high amount of variability in degrees of network

$$\equiv$$

Empirical degree distribution is heavy-tailed

Inhomogeneity increases if the deg. dist has diverging lower moments

Does inhomogeneity lead to fundamentally different behavior?

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Does inhomogeneity lead to fundamentally different behavior?

Yes, with respect to all above aspects (Q1)–(Q3)

Effect of inhomogeneity increases $\downarrow$

Finite third moment

Infinite third moment but finite second moment

Infinite second moment but finite first moment

Role of inhomogeneity: Three universality classes

Critical percolation on configuration model

Finite third moment

- ▷ Similar behavior as homogeneous models (Erdős-Rényi, Regular graphs)
- ▷ Insensitivity to the degree distribution

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- Deleting the highest degree vertex changes the scaling limits
- Comp sizes, distances crucially depend on the exact deg. distribution

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Infinite second moment but finite first moment

- ▷ $p_c = 0$: almost all edges must be deleted to remove the giant
- ▷ Critical behavior changes depending on multigraphs or simple graphs
  - ▶ Configuration model versus erased configuration model

Random graph: Configuration model

Set up: Power-law degrees (proportion of vertices of degree $k \approx Ck^{-\tau}$)

Finite third moment: $\tau > 4$, **Infinite third moment:** $\tau \in (3, 4)$,
Infinite second moment: $\tau \in (2, 3)$

Hubs: Vertices of degree Θ(max degree)

Preliminaries

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$C_{(i)}$:= the i-th largest component

$SP(C_{(i)})$:= # surplus edges in $C_{(i)}$

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Topology: $\mathbb{U}^0 \subset \mathbb{R}_+^\infty \times \mathbb{N}^\infty$ with norm $(\sum_i x_i^2)^{1/2} + \sum_i x_i y_i$

Component size and surplus (Q1)

Theorem 1 (D, v/d Hofstad, v Leeuwaarden, Sen '16 a,b)

For $\tau > 4$ and $p = p_c(1 + \lambda n^{-\frac{1}{3}})$

$$(n^{-\frac{2}{3}}|C_{(i)}|, \mathrm{SP}(C_{(i)}))_{i\geqslant 1} \xrightarrow{d} \mathbf{X}_1 \quad \text{in } \mathbb{U}^0 \qquad \text{(Finite third moment)}$$

For $\tau \in (3, 4)$ and $p = p_c(1 + \lambda n^{-\frac{\tau-3}{\tau-1}})$

$$(n^{-\frac{\tau-2}{\tau-1}}|C_{(i)}|, \mathrm{SP}(C_{(i)}))_{i\geqslant 1} \xrightarrow{d} \mathbf{X}_2 \quad \text{in } \mathbb{U}^0 \qquad \text{(Infinite third moment)}$$

- Generalizes Nachmias & Peres '09 (d-regular), Riordan '12 (bounded degree)
- $\mathbf{X}_1 \equiv$ Erdős-Rényi (insensitive to the exact degree distribution)
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$$\boxed{\lim_{n\to\infty} \mathbb{P}(\text{two hubs are in the same component}) \begin{cases} = 0, & \text{for } \tau > 4 \\ \in (0,1), & \text{for } \tau \in (3,4) \end{cases}}$$

Evolution of components as λ increases

$$
\mathbf{Z}_n(\lambda) := \begin{cases} \left(n^{-\frac{2}{3}}|C_{(i)}|, \mathrm{SP}(C_{(i)})\right)_{i\geqslant 1} & \tau > 4 \\ \\ \left(n^{-\frac{\tau-2}{\tau-1}}|C_{(i)}|, \mathrm{SP}(C_{(i)})\right)_{i\geqslant 1} & \tau \in (3,4) \end{cases}
$$

For ERRG,

- $C_{(i)}$ and $C_{(j)}$ merge at rate $|C_{(i)}| \times |C_{(j)}|$ (multiplicative coalescent)
- $\mathrm{SP}(C_{(i)})$ increases by 1 at rate $\binom{|C_{(i)}|}{2}$ (augmented version)

Previous works by [Aldous '97], [Aldous & Limic '98], [Bhamidi, Budhiraja & Wang '14], [Broutin & Marckert '16]

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Theorem 2 (D, v/d Hofstad, v Leeuwaarden, Sen '16 a,b)

$$(Z_n(\lambda))_{-\infty<\lambda<\infty} \xrightarrow{d} (\mathrm{AMC}_1(\lambda))_{-\infty<\lambda<\infty} \quad \text{in } (\mathbb{U}^0)^k \qquad \text{(Finite third moment)}$$

$$(Z_n(\lambda))_{-\infty<\lambda<\infty} \xrightarrow{d} (\mathrm{AMC}_2(\lambda))_{-\infty<\lambda<\infty} \quad \text{in } (\mathbb{U}^0)^k \qquad \text{(Infinite third moment)}$$

Convergence of distances (Q3)

Seminal work by Addario-Berry, Broutin, Goldschmidt (2012)

- $\mathcal{C}_{(i)}$ as a random metric space
  - Elements: vertices in $\mathcal{C}_{(i)}$
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Metric structure of critical components

Theorem 3 (Bhamidi, Broutin, Sen, Wang '14) (Bhamidi, Sen '16)

Re-scale metric by $n^{-\frac{1}{3}}$. Let $\tau > 4$ and $p = p_c(1 + \lambda n^{-\frac{1}{3}})$. Then

$$(\mathcal{C}_{(i)})_{i \geqslant 1} \text{ converges in distribution}$$

Theorem 4 (Bhamidi, D, v/d Hofstad, Sen '17, '18+)

Re-scale metric by $n^{-\frac{\tau-3}{\tau-1}}$. Let $\tau \in (3, 4)$ and $p = p_c(1 + \lambda n^{-\frac{\tau-3}{\tau-1}})$. Then

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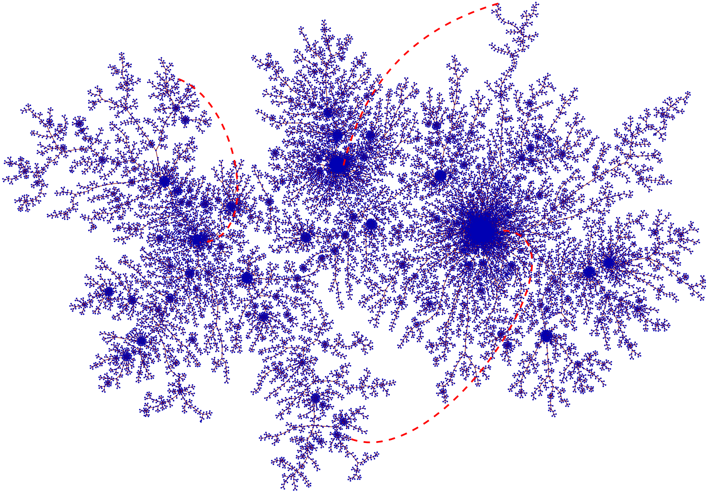
Limiting object: infinite third moment



surplus = Poisson $\qquad$ distances $= n^{\frac{\tau-3}{\tau-1}}$ $\qquad$ size $= n^{\frac{\tau-2}{\tau-1}}$

Courtesy: Igor Kortchemski's website

Infinite second moment case

Degree distribution: Power-law $\sim Ck^{-\tau}$ with $\tau \in (2, 3)$

- These networks are always robust
- $p > 0$: Always supercritical $\implies \boxed{p_c = 0}$

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- ▷ Analyze near critical behavior

$$\boxed{\text{Component sizes concentrate outside critical window}}$$

All of these were open questions till date...

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p_c	$\lambda n^{-\frac{3-\tau}{\tau-1}}$	$\lambda n^{-\frac{3-\tau}{2}}$
$\lvert C_{(i)}\rvert$	$n^{\frac{\tau-2}{\tau-1}}$	$n^{\frac{1}{\tau-1}-\frac{3-\tau}{2}}$

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- Single-edge constraint changes critical value for $\tau \in (2,3)$!
- ECM has larger component sizes and critical value
- In both cases, critical window is precisely the value when

$$\boxed{\lim_{n\to\infty} \mathbb{P}(\text{hubs are in the same component}) = \zeta \in (0,1)}$$

Subcritical regime: $\mathbb{P}(\text{hubs are in the same component}) \to 0$

Supercritical regime: $\mathbb{P}(\text{hubs are in the same component}) \to 1$

Configuration model results

Theorem 5 (D, v/d Hofstad, v Leeuwaarden '18+)

For $p_c = \lambda n^{-\frac{3-\tau}{\tau-1}}$:

$$(n^{-\frac{\tau-2}{\tau-1}}|\mathcal{C}_{(i)}|, \mathrm{SP}(\mathcal{C}_{(i)}))_{i\geqslant 1} \xrightarrow{d} (|\gamma_i|, N(\gamma_i))_{i\geqslant 1}$$

in $\mathbb{U}^0_{\downarrow}$ topology, where $(\gamma_i)_{i\geqslant 1}$ is the ordered excursions of

$$S_\infty(t) = \lambda \sum_{i=1}^{\infty} i^{-\frac{1}{\tau-1}} \mathbb{1}_{\{\mathrm{Exp}(1/i^{\frac{1}{\tau-1}}\mu)\leqslant t\}} - 2t, \quad N(\gamma_i) = \text{Poisson}(\text{area under } \gamma_i)$$

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Theorem 6 (D, v/d Hofstad, v Leeuwaarden '18+)

For $p \ll p_c = \lambda n^{-\frac{3-\tau}{\tau-1}}$: $(n^{\frac{1}{\tau-1}}p)^{-1}|\mathcal{C}_{(i)}| \xrightarrow{\mathbb{P}} ci^{-\frac{1}{\tau-1}}$

For $p \gg p_c = \lambda n^{-\frac{3-\tau}{\tau-1}}$: $(np^{\frac{1}{3-\tau}})^{-1}|\mathcal{C}_{(1)}| \xrightarrow{\mathbb{P}} c, \quad |\mathcal{C}_{(2)}| \ll |\mathcal{C}_{(1)}|$

Erased configuration model results

Theorem 7 (Bhamidi, D, v/d Hofstad, v Leeuwaarden '18+)

For $p \ll p_c = \lambda n^{-\frac{3-\tau}{2}}$: $(n^{\frac{1}{\tau-1}} p)^{-1} |C_{(i)}| \xrightarrow{\mathbb{P}} c i^{-\frac{1}{\tau-1}}$

Theorem 8 (Bhamidi, D, v/d Hofstad, v Leeuwaarden '18+)

For $p = p_c = \lambda n^{-\frac{3-\tau}{2}}$, $\lambda \in (0, \lambda_0)$: in $\ell^2_{\downarrow}$ topology

$$\left((n^{\frac{1}{\tau-1}} p_c)^{-1} |C_{(i)}|\right)_{i \geqslant 1} \xrightarrow{d} (W_i^{\infty})_{i \geqslant 1}$$

Limit object: $G_\infty(\lambda)$ is a graph on $\mathbb{Z}_+$, where vertices i and j share Poisson(λ_{ij}) edges with λ_{ij}

$$\lambda_{ij} := \lambda^2 \int_0^\infty \Theta_i(x)\Theta_j(x) dx, \quad \Theta_i(x) := \frac{c_F^2 i^{-\alpha} x^{-\alpha}}{\mu + c_F^2 i^{-\alpha} x^{-\alpha}}$$

W_i^∞ is the i-th largest value of

$$\left\{ \sum_{i \in C} i^{-\alpha} : C \text{ is a connected component of } G_\infty(\lambda) \right\}$$

Erased configuration model results

Theorem 7 (Bhamidi, D, v/d Hofstad, v Leeuwaarden '18+)

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Theorem 9 (Bhamidi, D, v/d Hofstad, v Leeuwaarden '18+)

For $p = p_c = \lambda n^{-\frac{3-\tau}{2}}$, $\lambda > \lambda_0$: $\mathbb{P}$(all hubs in same component) $\to 1$

Summary

Critical percolation on configuration model $\tau =$ power-law exponent

Three fundamental questions:

(Q1) Component size and surplus (at each fixed λ)

(Q2) Evolution of component size and surplus over critical window (as λ varies)

(Q3) Graph distances within components (at each fixed λ)

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- $\tau \in (3, 4)$: Deleting the highest degree vertex changes scaling limits
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Thank you