On global in time solutions for two-fluid interfaces.

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Two scenarios

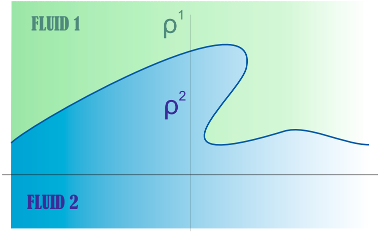


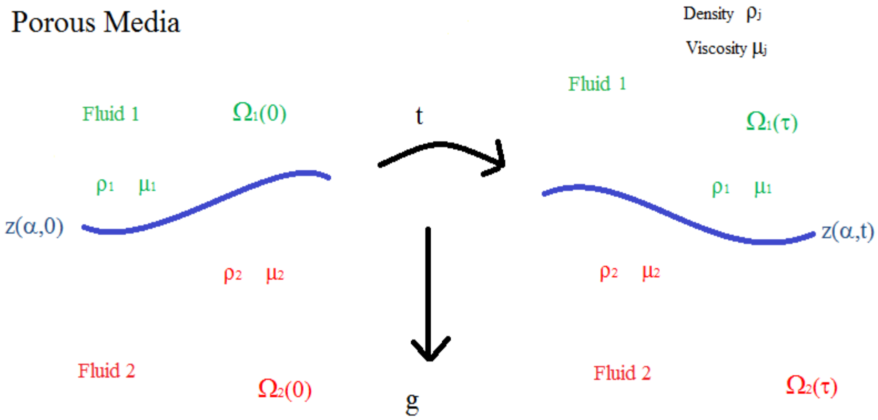
Figure: Two fluids: internal waves

- Muskat equation Global existence with arbitrarily large slope.

  with O. Lazar. arXiv 2018
- Euler equations Stationary solutions with a splash singularity

  with A. Enciso and N. Grubic. To appear in arXiv

The Muskat problem (Muskat (1934), Saffman & Taylor (1958))



In this talk:

- Scenario in $\mathbb{R}^2$
- Finite energy
- No surface tension
- $\mu_1 = \mu_2$

We consider:

1. Open curves vanishing at infinity

$$\lim_{\alpha\to\infty} (z(\alpha,t) - (\alpha,0)) = 0,$$

2. Periodic curves in the space variable

$$z(\alpha + 2k\pi, t) = z(\alpha,t) + 2k\pi(1,0).$$

3. Closed curves $\Rightarrow$ Unstable regime.

Incompressible porous media equation

Two-dimensional mass balance equation in porous media (2D IPM)

$$
\left\{ \boxed{\begin{array}{l} \rho_t + u \cdot \nabla \rho = 0 \\ \frac{\mu}{\kappa} u = -\nabla p - (0, g\rho) \\ \operatorname{div} u = 0 \end{array}} \right.
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Remark: let $\mu = \kappa = g = 1$

- $u(x) = \frac{1}{2\pi} PV \int_{\mathbb{R}^2} (-2\frac{y_1 y_2}{|y|^4}, \frac{y_1^2 - y_2^2}{|y|^4})\, \rho(x-y) dy \; - \frac{1}{2}\, (0, \rho(x))\,,$
- $||\rho||_{L^p}(t) = ||\rho||_{L^p}(0) \quad p \in [1, \infty] \Longrightarrow ||u||_{L^p}(t) \leq C \quad p \in (1, \infty)$
- $(\partial_t + u \cdot \nabla)\nabla^\perp \rho = (\nabla u)\nabla^\perp \rho.$

Muskat: Contour equation

We consider

$$\rho(x,t) = \begin{cases} \rho^1 & x \in \Omega^1(t) \\ \rho^2 & x \in \Omega^2(t) \end{cases}$$

with

$$\partial\Omega^j(t) = \{z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)) \,:\, \alpha \in \mathbb{R}\}.$$

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Darcy's law:

$$u = -\nabla p - (0,\rho) \Rightarrow \nabla^\perp \cdot u = -\partial_{x_1}\rho.$$

$$\nabla^\perp \cdot u(x,t) = -(\rho^2 - \rho^1)\partial_\alpha z_2(\alpha,t)\delta(x - z(\alpha,t)).$$

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Biot-Savart:

$$u(x,t) = -\frac{\rho^2 - \rho^1}{2\pi} PV \int_{\mathbb{R}} \frac{(x - z(\beta,t))^{\perp}}{|x - z(\beta,t)|^2} \partial_\alpha z_2(\beta,t) d\beta,$$

for $x \neq z(\alpha,t)$.

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for $x \neq z(\alpha,t)$.

$$\|u\|_{L^2}(t) < \infty.$$

Muskat: Contour equation

It yields

$$\boxed{z_t(\alpha) = \frac{\rho^2-\rho^1}{2\pi} PV \int_{\mathbb{R}} \frac{z_1(\alpha)-z_1(\beta)}{|z(\alpha)-z(\beta)|^2} (\partial_\alpha z(\alpha) - \partial_\beta z(\beta)) d\beta.}$$

Contour equation as a graph

- The equation for a graph $z(\alpha, t) = (\alpha, f(\alpha, t))$.

$$\alpha_t = \frac{\rho^2 - \rho^1}{2\pi} \int_{\mathbb{R}} \frac{(\alpha - \beta)(\partial_\alpha \alpha - \partial_\beta \beta)}{(\alpha - \beta)^2 + (f(\alpha) - f(\beta))^2} d\beta$$

$$(0 = \quad 0)$$

$$\boxed{f_t(\alpha) = \frac{\rho^2 - \rho^1}{2\pi} \int_{\mathbb{R}} \frac{(\alpha - \beta)(\partial_\alpha f(\alpha) - \partial_\beta f(\beta))}{(\alpha - \beta)^2 + (f(\alpha) - f(\beta))^2} d\beta}$$

with initial data

$$z_1(\alpha, 0) = \alpha$$

$$z_2(\alpha, 0) = f(\alpha, 0) = f_0(\alpha).$$

The linearized equation

$$f_t^L(\alpha,t) = -\frac{\rho^2-\rho^1}{2}\Lambda(f^L)(\alpha,t), \qquad \Lambda = (-\Delta)^{1/2}.$$

Fourier transform:

$$\widehat{f^L}(\xi,t) = \widehat{f_0}(\xi,t)\exp\Big(-\frac{\rho^2-\rho^1}{2}|\xi|t\Big).$$

- $\rho^2 > \rho^1$ stable case,
- $\rho^2 < \rho^1$ unstable case.

Local existence theory

For a general interface

$$\partial\Omega^j(t) = \{\mathbf{z}(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)), \quad \alpha \in \mathbb{R}\}$$

after taking k derivatives $(k \geq 3)$ it can be shown that

$$\partial_t \partial_\alpha^k \mathbf{z}(\alpha,t) = -\underbrace{(\rho^2 - \rho^1)\frac{\partial_\alpha z_1(\alpha,t)}{|\partial_\alpha \mathbf{z}(\alpha,t)|^2}}_{\sigma(\alpha,t)\equiv R-T} \Lambda \partial_\alpha^k \mathbf{z}(\alpha,t) + \text{l.o.t.}$$

Thus we can distinguish three regimes:

- Stable regime: $\sigma > 0 \implies$ the denser fluid is always below.
  The Muskat problem is locally well-posed in time in Sobolev's spaces.
- Fully unstable regime: $\sigma < 0 \implies$ the denser fluid is always above.
  The Muskat problem is ill-posed in Sobolev's spaces.
- Partial unstable regime: σ has not a defined sign $\implies$ there is a part of the interface where the denser fluid is above.

Energy estimates for the stable regime $\rho^2 > \rho^1$

For $k = 3$:

$$\frac{d}{dt}||f||_{H^3}^2 = -\int \sigma(\alpha)\partial_\alpha^3 f(\alpha)\Lambda\partial_\alpha^3 f(\alpha)d\alpha + \text{Controlled Quantities}$$

Then, since $\sigma > 0$, yields

$$\begin{aligned} -\int \sigma(\alpha)\partial_\alpha^3 f(\alpha)\Lambda\partial_\alpha^3 f(\alpha) &\leq -\frac{1}{2}\int \sigma(\alpha)\Lambda\left(\partial_\alpha^3 f(\alpha)\right)^2 d\alpha \\ &\leq -\frac{1}{2}\int \Lambda\sigma(\alpha)\left(\partial_\alpha^3 f(\alpha)\right)^2 \end{aligned}$$

Finally we obtain

$$\frac{d}{dt}||f||_{H^3}^2 \leq C||f||_{H^3}^m$$

Local existence results in the stable regime

- D.C. and F. Gancedo (2007). Local existence in H^3 (and ill-posedness for $\rho^2 < \rho^1$).
- A. Cheng, R. Granero and S. Shkoller (2016). Local existence in H^2.
- P. Constantin, F. Gancedo, R. Shvydkoy and V. Vicol (2017). Local existence in $W^{2,p}$ for p>1.
- B-V. Matioc (arxiv). Local existence in $H^{\frac{3}{2}+\epsilon}$.

Conserved quantities in the stable regime: $(z_1, z_2) = (\alpha, f(\alpha, t))$

- $\int f(\alpha, t)\, d\alpha = \int f_0(\alpha)\, d\alpha.$

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- Maximum principle for the L^2-norm

$$||f(\cdot, t)||^2_{L^2(\mathbb{R})} + \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \log\left(1 + \left(\frac{f(\alpha) - f(\beta)}{\alpha - \beta}\right)^2\right) d\alpha d\beta = ||f_0||^2_{L^2(\mathbb{R})}$$

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Compare with the linear case

$$||f(\cdot,t)||_{L^2(\mathbb{R})} + \int_0^T \underbrace{\int_{\mathbb{R}} \left(\frac{f(\alpha) - f(\beta)}{\alpha - \beta}\right)^2 d\alpha d\beta}_{=\int_{\mathbb{R}} f(x)\Lambda f(x) dx = ||\Lambda^{\frac{1}{2}} f(\cdot,t)||^2_{L^2(\mathbb{R})}} dt = ||f_0||^2_{L^2(\mathbb{R})}$$

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But

$$\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \log\left(1 + \left(\frac{f(\alpha)-f(\beta)}{\alpha-\beta}\right)^2\right) d\alpha d\beta \leq C||f(\cdot,t)||_{L^1}$$

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- Maximum principle: $||f||_{L^\infty}(t) \leq ||f||_{L^\infty}(0)$.

  Periodic case:

$$||f - \frac{1}{2\pi}\int_{\mathbb{T}} f_0 d\alpha||_{L^\infty}(t) \leq ||f_0 - \frac{1}{2\pi}\int_{\mathbb{T}} f_0 d\alpha||_{L^\infty} e^{-Ct}.$$

  Flat at infinity: $||f||_{L^\infty}(t) \leq \dfrac{||f_0||_{L^\infty}}{1 + Ct}$.

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- Maximum principle: If $\|f_\alpha\|_{L^\infty}(0) < 1$ then $\|f_\alpha\|_{L^\infty}(t) \leq \|f_\alpha\|_{L^\infty}(0)$.

Global existence for $||\partial_\alpha f_0||_{L^\infty(\mathbb{R})} < 1$

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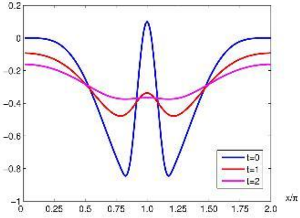
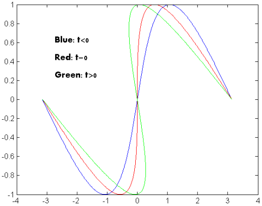
- S. Cameron (arxiv2017). Global classical solutions if $||\partial_\alpha f_0||_{L^\infty(\mathbb{R})} < 1$.

What happens if $||\partial_\alpha f_0||_{L^\infty(\mathbb{R})} > 1$ (with finite energy)?

- Numerical simulations of Turning (i.e. shift of stability) by Maria López-Fernández





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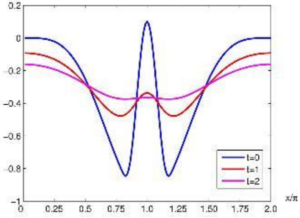

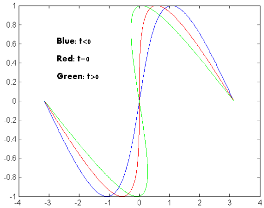

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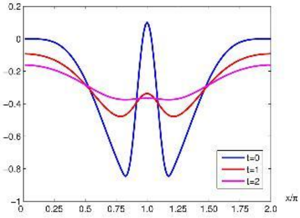

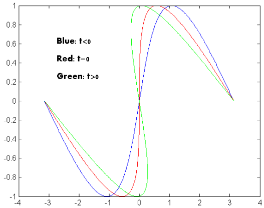

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- Numerical evidence of turning with $||\partial_\alpha f_0||_{L^\infty} = 22$ by J. Gómez-Serrano.
  Is there a turning for $||\partial_\alpha f_0||_{L^\infty} = 1 + \epsilon$?

What happens after Turning?

- In the stable regime a solution of Muskat becomes immediately real-analytic and then passes to the unstable regime in finite time. Moreover, the Cauchy-Kowalewski theorem shows that a real-analytic Muskat solution continues to exist for a short time after the turnover.

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- Breakdown of smoothness: There exist interfaces of the Muskat problem such that after turnover their smoothness breaks down (is not C^4). Joint work with A. Castro, C. Fefferman and F. Gancedo.
- Double shift of stability: Turning stable-unstable-stable (also unstable-stable-unstable). Joint work with J. Gómez-Serrano and A. Zlatos.

Global existence for arbitrarily large slope

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Theorem

Assume $f_0 \in H^{5/2}$ with $\|f_0\|_{\dot{H}^{3/2}}$ small enough, then, there exists a unique strong solution f which verifies $f \in L^\infty([0,T], H^{3/2}) \cap L^2([0,T], \dot{H}^{5/2})$, for all $T > 0$.

Joint work with O. Lazar.

Global existence for arbitrarily large slope: proof

Main steps of the proof:

- The proof is based on the use of a new formulation of the Muskat equation that involves oscillatory terms as well as a careful use of Besov space techniques.
- $$f_t(t,x) = \frac{\rho}{\pi} P.V. \int \partial_x \Delta_\alpha f \int_0^\infty e^{-\delta} \cos(\delta \Delta_\alpha f)\, d\delta\, d\alpha$$
  $$f(0,x) = f_0(x).$$

  where $\Delta_\alpha f \equiv \frac{f(x,t) - f(x-\alpha,t)}{\alpha}$.

Global existence for arbitrarily large slope: proof

- A priori estimates in $\dot{H}^{3/2}$:

$$\begin{aligned}\frac{1}{2}\partial_t\|f\|^2_{\dot{H}^{3/2}} &= \int \mathcal{H}f_{xx}\int \partial_{xx}\Delta_\alpha f \int_0^\infty e^{-\delta}\cos(\delta\Delta_\alpha f(x))\,d\delta\,d\alpha\,dx \\ &- \int \mathcal{H}f_{xx}\int (\partial_x\Delta_\alpha f)^2 \int_0^\infty \delta e^{-\delta}\sin(\delta\Delta_\alpha f(x))\,d\delta\,d\alpha\,dx \\ &= I_1 + I_2\end{aligned}$$

We can estimate

$$|I_2| \leq \|f\|^2_{\dot{H}^2}\|f\|_{\dot{H}^{3/2}}$$

and the most singular term is I_1

$$|I_1| \lesssim \|f\|^2_{H^2}(\|f\|^2_{\dot{H}^{3/2}} + \|f\|_{\dot{H}^{3/2}}) - \pi\|f\|^2_{\dot{H}^2} + \pi\frac{K^2}{1+K^2}\|f\|^2_{\dot{H}^2}$$

where $K = \|f_x\|_{L^\infty L^\infty}$.

Global existence for arbitrarily large slope: proof

- A priori estimates in $\dot{H}^{3/2}$:

$$\begin{aligned}\frac{1}{2}\partial_t\|f\|^2_{\dot{H}^{3/2}} &= \int \mathcal{H}f_{xx}\int \partial_{xx}\Delta_\alpha f\int_0^\infty e^{-\delta}\cos(\delta\Delta_\alpha f(x))\,d\delta\,d\alpha\,dx\\ &- \int \mathcal{H}f_{xx}\int (\partial_x\Delta_\alpha f)^2\int_0^\infty \delta e^{-\delta}\sin(\delta\Delta_\alpha f(x))\,d\delta\,d\alpha\,dx\\ &= I_1 + I_2\end{aligned}$$

We can estimate

$$|I_2| \leq \|f\|^2_{\dot{H}^2}\|f\|_{\dot{H}^{3/2}}$$

and the most singular term is I_1

$$|I_1| \lesssim \|f\|^2_{\dot{H}^2}(\|f\|^2_{\dot{H}^{3/2}} + \|f\|_{\dot{H}^{3/2}}) - \pi\|f\|^2_{\dot{H}^2} + \pi\frac{K^2}{1+K^2}\|f\|^2_{\dot{H}^2}$$

where $K = \|f_x\|_{L^\infty L^\infty}$.

- Then

$$\frac{1}{2}\partial_t\|f\|^2_{\dot{H}^{3/2}} + \frac{\pi}{1+K^2}\|f\|^2_{\dot{H}^2} \leq C\|f\|^2_{\dot{H}^2}\left(\|f\|^2_{\dot{H}^{3/2}} + \|f\|_{\dot{H}^{3/2}}\right)$$

Global existence for arbitrarily large slope: proof

- Similar a priori estimates in $\dot{H}^{5/2}$:

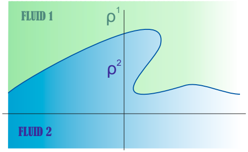
Lemma

Let $T > 0$ and $f_0 \in \dot{H}^{5/2} \cap \dot{H}^{3/2}$ so that $\|f_0\|_{\dot{H}^{3/2}} < C(\|f_{0,x}\|_{L^\infty})$, then we have

$$\begin{aligned}\|f\|^2_{\dot{H}^{5/2}}(T) \quad &+ \quad \frac{\pi}{1+M^2}\int_0^T \|f\|^2_{\dot{H}^3}\, ds \\ &\lesssim \quad \|f_0\|_{\dot{H}^{5/2}} + \left(\|f\|_{L^\infty([0,T],\dot{H}^{3/2})} + \|f\|^2_{L^\infty([0,T],\dot{H}^{3/2})}\right)\int_0^T \|f\|^2_{\dot{H}^3}\, ds\end{aligned}$$

where M is the space-time Lipschitz norm of f.

Two fluids. Euler equations



In each domain, the fluid flow is governed by the incompressible, irrotational Euler equations; that is, the respective velocities u^j and the corresponding pressures p^j satisfy

$$\rho_j(\partial_t u^j + u^j \cdot \nabla) u^j = -\nabla p^j - g\rho_j e_2 \quad \text{in} \quad \Omega_j, \tag{1a}$$

$$\nabla \cdot u^j = 0 \quad \text{and} \quad \nabla^\perp u^j = 0 \quad \text{in} \quad \Omega_j, \tag{1b}$$

$$p^1 - p^2 = -\sigma K \quad \text{on} \quad \partial\Omega \tag{1c}$$

$$(\partial_t z - u^j) \cdot (\partial_\alpha z)^\perp = 0 \quad \text{on} \quad \partial\Omega, \tag{1d}$$

where $\partial\Omega(t) = \{z(\alpha,t) = (z_1(\alpha,t), z_2(\alpha,t)) \,:\, \alpha \in \mathbb{R}\}$.

Free boundary problem: one fluid

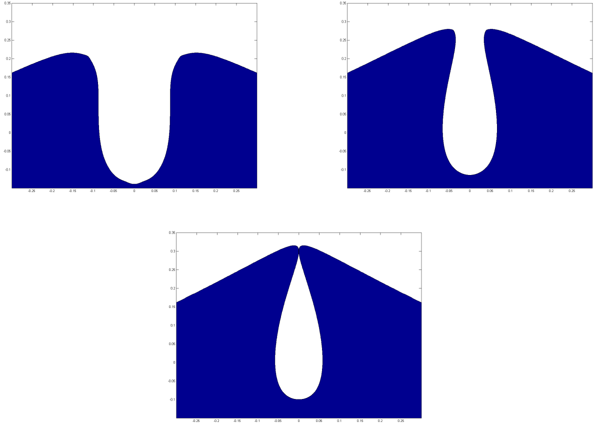
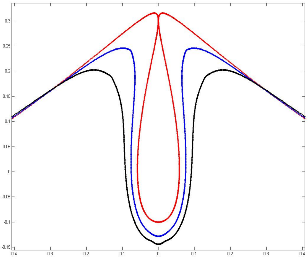


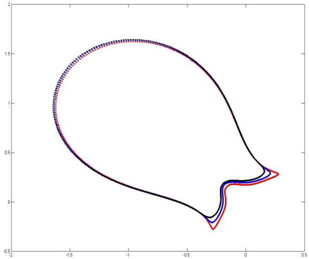
Figure: Turnover and a Splash singularity.

A. Castro, D.C., C.Fefferman, F. Gancedo and J. Gómez-Serrano (2011)

Main ideas of the proof of the Splash

- The water wave equations are invariant under time reversal.
- We can choose initially the normal component of the velocity on the interface.
- Solving the equations backwards in time (prove local existence).





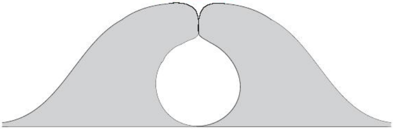
$$P(w) = \left(\tan\left(\frac{w}{2}\right)\right)^{1/2}, \quad w \in \mathbb{C},$$

Further Results

- Splat

A Variant of the Splash:

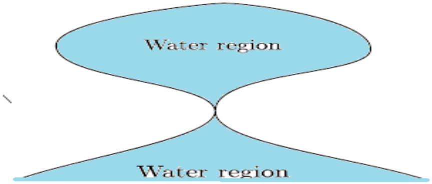
SPLAT!



At time t_2, the interface self-intersects along an arc, but u and $\partial\Omega$ are otherwise smooth.

- Surface tension
- Non trivial vorticity by D. Coutand and S. Shkoller (2012)
- Viscosity (see also D. Coutand and S. Shkoller)

Squeezing a fluid



- Can we squeeze an incompressible fluid?

Two fluids

- Two incompressible fluids with nonzero densities cannot form a splash.
  Ch. Fefferman, A. Ionescu, V. Lie (see also D. Coutand and S. Shkoller)

Sketch of Proof: Consider the vorticity $\nabla \times u(x,t) = \omega(\alpha,t)\delta(x - z(\alpha,t))$.

- If

$$|\partial_\alpha^k z(\alpha,t)| \quad (k = 0,1,2,3,4)$$

  and

$$max\{|\partial_x^\beta u^I(x,t)| : x \in \Omega^I(t), |\beta| \leq 3$$

  remain bounded as $t \to T_*$, then $|\omega(\alpha,t)|$ remains bounded as $t \to T_*$, because ω satisfies a variant of Burgers equation.

- If $|\omega(\alpha,t)|$ remains bounded as $t \to T_*$, then, because the interface moves with the fluid, the function $F(t) = \frac{1}{C_{CA(t)}}$ satisfies

$$|\frac{dF}{dt}| \leq Const.|F|ln(|F|+2)$$

  Hence, F(t) remains bounded as $t \to T_*$, so the splash cannot form.

Stationary Splash solutions with two fluids

- Theorem: Let us fix the density of the second fluid $\rho_2 > 0$. Then for any sufficiently small upper fluid density $\rho_1 > 0$ and $g > 0$, there is some positive surface tension coefficient for which there exists a stationary solution two-fluid Euler equations such that the interface $\partial\Omega$ has a Splash singularity. The regularity of $\partial\Omega$ and ω is $C^{2,\alpha}$ and C^{α} with $0 < \alpha < \frac{1}{2}$.

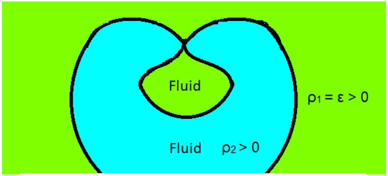


Figure: Two fluids: internal waves

The idea of the proof is to perturb a family of exact stationary water waves introduced by Crapper (1957). Joint work with A. Enciso and N. Grubic.

Stationary solutions

In each domain, the fluid flow is governed by the stationary, incompressible, irrotational Euler equations; that is, the respective velocities v^j and the corresponding pressures p^j satisfy

$$\rho_j(v^j \cdot \nabla)v^j = -\nabla p^j - g\rho_j e_2 \quad \text{in} \quad \Omega_j, \tag{2a}$$
$$\nabla \cdot v^j = 0 \quad \text{and} \quad \nabla^\perp v^j = 0 \quad \text{in} \quad \Omega_j, \tag{2b}$$
$$v^j \cdot (\partial_\alpha z)^\perp = 0 \quad \text{on} \quad \partial\Omega, \tag{2c}$$
$$p^1 - p^2 = -\sigma K \quad \text{on} \quad \partial\Omega. \tag{2d}$$

We assume that the interface satisfies periodicity conditions

$$z_1(\alpha + 2\pi) = z_1(\alpha) + 2\pi, \quad z_2(\alpha + 2\pi) = z_2(\alpha)$$

and is symmetric with respect to the y-axis:

$$z_1(-\alpha) = -z_1(\alpha), \quad z_2(-\alpha) = z_2(\alpha).$$

Stationary solutions

To fix the parametrization, we use the hodograph transform with respect to the lower fluid. Then, as song as there is no self-intersections: A stationary solution of the two-fluid system is reduced to finding 2π-periodic functions $\omega(\alpha)$ and $z(\alpha) - (\alpha, 0)$ satisfying

$$2|\partial_\alpha z|^2 M(z) + \epsilon\,\omega(\omega - 2) = 2, \tag{3a}$$

$$2BR(z,\omega) \cdot \partial_\alpha z + \omega = 2, \tag{3b}$$

$$BR(z,\omega) \cdot \partial_\alpha^\perp z = 0, \tag{3c}$$

where BR and M are given by

$$BR(z,\omega) = \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(z(\alpha,t) - z(\beta,t))^\perp}{|z(\alpha,t) - z(\beta,t)|^2} \omega(\alpha,t) d\alpha$$

$$M(z) = -\frac{2\rho_2}{\rho_2 - \rho_1} q K(z) - 2g z_2 + 1,$$

$q := \frac{\sigma}{\rho_2}$, $\epsilon := \frac{2\rho_1}{\rho_2 - \rho_1}$, $K(z)$ is the curvature of the interface.

Stationary solutions when $\epsilon = 0$ and $g = 0$

- The system decouples and we recover the pure capillary waves (Levi-Civita 1925).
- This problem admits a family of exact solutions depending on the parameter q. In fact, Crapper has shown that the family of functions

$$z_A(\alpha) = \alpha + \frac{4i}{1 + Ae^{-i\alpha}} - 4i.$$

are solutions. Parameter A depends on q via

$$q = \frac{1 + A^2}{1 - A^2}.$$

and it actually suffices to consider $A \geq 0$, since the transformation $A \mapsto -A$ corresponds to a translation $\alpha \to \alpha + \pi$.

- We solve for ω by inverting

$$2BR(z_A, \omega) \cdot \partial_\alpha z_A + \omega = 2$$

Stationary solutions

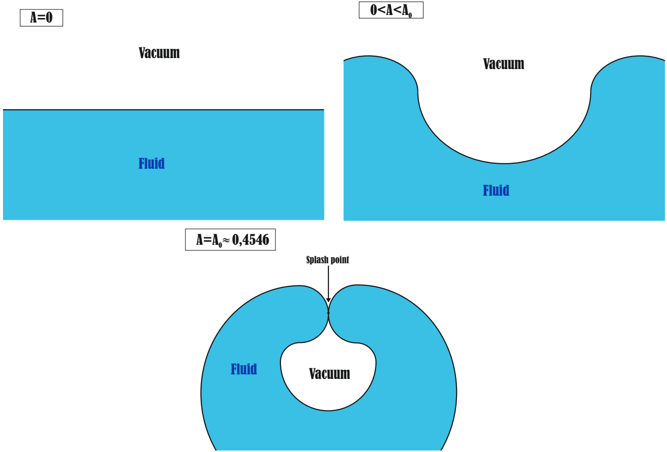


Figure: Interface at different values of the parameter A.

For $A = A_0$, the curve $z_A(\alpha)$ exhibits a splash, while for A slightly larger than A_0 the curve intersects at exactly two points, and the intersection is transverse.

Invert the operator $\omega + 2BR(z,\omega)\cdot\partial_\alpha z$

- G. Baker, D. Meiron and S. Orszag (1982): Let $z \in H^3$ and assume z is a curve without self-intersections. Then $\mathcal{A}(z)(\omega) = 2BR(z,\omega)\cdot\partial_\alpha z$ defines a compact linear operator

$$\mathcal{A}(z) : H^1 \to H^1$$

  whose adjoint T^*, acting on ω, is described in terms of the Cauchy integral of ω along the curve z whose eigenvalues are strictly smaller than 1 in absolute value. In particular, the operator $1 + \mathcal{A}(z)$ is invertible.

- A. Cordoba, D.C. and F. Gancedo (2010): Control of the norm of the inverse operators $(I - \eta\mathcal{A}(z))^{-1}$, $|\eta| \leq 1$ in terms of the chord-arc condition and the regularity of z. The arguments rely upon the boundedness properties of the Hilbert transforms associated to $C^{1,\alpha}$ curves, for which we need precise estimates obtained with arguments involving conformal mappings, Hopf maximum principle and Harnack inequalities.

Results

By the Implicit function Theorem

- There are almost-splash stationary solutions to the Euler equations with two fluids.
- The existence of stationary splash singularities for one fluid.

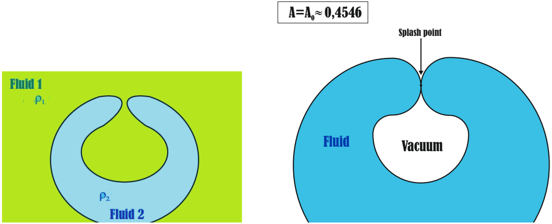
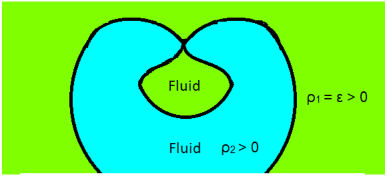


Figure: At times t=0 and t=T>0.

In the case of two fluids



- When the Chord-arc fails there is difficulties to invert the operator within the framework of the usual Sobolev spaces.
- V. Maz'ya developed techniques to treat the case of cusp domains within the class of weighted spaces depending on the order of the cusp μ where the interface approaches the splash point (cusp tip $x = 0$) as $(x, x^{1+\mu})$ with $\mu > 0$.

In the case of two fluids

Let

$$w_\beta(x) := |x|^\beta$$

be the weight function for $\beta \in \mathbb{R}$ and x in some interval $I \in \mathbb{R}$ containing the origin. Then for $k \in \mathbb{N}$

$$u \in W^k_{p,\beta} \;:\Longleftrightarrow\; w_{\beta+j}(x)\partial^j_x u \in L^p,\, j \le k.$$

We adapted Maz'ya technique to show that $1 + \mathcal{A}(z)$ actually has values in a smaller Banach space; i.e. we show

- $1 + \mathcal{A}(z) : W^1_{p,\beta} \to X_{\beta,\mu}$ continuous on a closed subspace $X_{\beta,\mu} \subset W^1_{p,\beta}$,
- $1 + \mathcal{A}(z) : W^1_{p,\beta} \to X_{\beta,\mu}$ invertible by using conformal maps.

Finally, after adjusting the Banach space for z, we show that we can use the implicit function theorem on the perturbed equations defined on these new weighted Sobolev spaces.

Motivation: work in progress

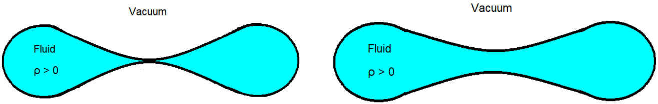


Figure: At times t=0 and t=T>0.

- Goal: to prove local existence starting from a Splash
- Obtain a priori estimates for a carefully chosen energy functional within the weighted Sobolev spaces.
- Choose an initial data that opens the Splash.

Joint work with A. Enciso, C. Fefferman and N. Grubic.

THANK YOU