Stokes systems with variable coefficients

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Abstract:

I will report some recent results about both stationary and non-stationary Stokes systems with variable coefficients. Applications to the Navier-Stokes equations and the construction of Green's functions will also be presented.

Based on joint work with Doyoon Kim (Korea University), Jongkeun Choi (Brown University), and Tuoc Phan (U of Tennessee).

The classical linear Stokes system:

$$u_t - \Delta u + \nabla p = f, \quad \operatorname{div} u = g,$$

where u is a vector-valued function, p is a scalar function, and f is the external force.

Relation to the Navier-Stokes equations.

1. We can view the nonlinear term as external force:

$$u_t - \Delta u + \nabla p = -u \cdot \nabla u, \quad \operatorname{div} u = 0.$$

2. We can also rewrite the equation into the form

$$u_t - D_i((\delta_{ij} + d_{ij})D_j u) + \nabla p = 0, \quad \operatorname{div} u = 0,$$

where d_{ij} is a skew-symmetric matrix-valued function and satisfies

$$\Delta d_{ij} = D_j u_i - D_i u_j.$$

Outline:

I. L_p and Dini type estimates for stationary Stokes systems.

II. A construction of Green matrices for stationary Stokes systems.

III. L_p estimates for non-stationary Stokes systems, with applications.

I. L_p and Dini type estimates for the stationary Stokes system.

The classical stationary Stokes system in smooth domains

$$\begin{cases} \Delta u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = g & \text{in } \Omega \end{cases}$$

with the non-homogeneous Dirichlet boundary condition $u = \varphi$ on $\partial\Omega$ was studied by Ladyženskaya (1959), Sobolevskiĭ (1960), Cattabriga (1961), Vorovič and Judovič (1961), and Amrouche and Girault (1991).

Theorem. (Cattabriga, 1961) Let Ω be a bounded C^2 domain in $\mathbb{R}^3$. Then

$$\|Du\|_{L_q(\Omega)} + \|p\|_{L_q(\Omega)} \le N\|f\|_{W_q^{-1}(\Omega)} + N\|g\|_{L_q(\Omega)} + N\|\varphi\|_{W_q^{1-1/q}(\partial\Omega)}.$$

The proof is based on the explicit representation of solutions using fundamental solutions.

- Amrouche and Girault (1991): bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^d$, for any $d \geq 2$.
  Their proof is based on a result by Agmon, Douglis, and Nirenberg (1964) for elliptic systems together with an interpolation argument.
- Galdi, Simader, and Sohr (1994): a bounded Lipschitz domain with a sufficiently small Lipschitz constant.
- Fabes, Kenig, and Verchota (1988): arbitrary Lipschitz domains with the range of q restricted, by using the layer potential method and Rellich identities.
- Subsequent work by Z. Shen, Brown-Shen, M. Mitrea-Taylor, M. Mitrea-Wright, Geng-Kilty.

We are interested in the Stokes system with variable coefficients:

$$\begin{cases} \mathcal{L}u + \nabla p = f + D_\alpha f_\alpha & \text{in } \Omega, \\ \operatorname{div} u = g & \text{in } \Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^d$ and $\mathcal{L}$ is a strongly elliptic operator, given by

$$\mathcal{L}u = D_\alpha \left(A^{\alpha\beta} D_\beta u \right), \quad A^{\alpha\beta} = [A_{ij}^{\alpha\beta}]_{i,j=1}^d$$

for $\alpha, \beta = 1, \ldots, d$.

Such type of systems were considered by Giaquinta and Modica (1982) for sufficiently regular.

Motivations:

a. inhomogeneous fluids with density dependent viscosity;

b. equations which describe flows of shear thinning and shear thickening fluids with viscosity depending on pressure;

c. Navier-Stokes system in general Riemannian manifolds.

Conditions.

The coefficients $A^{\alpha\beta}$ are bounded and satisfy the strong ellipticity condition, i.e., there exists a constant $\delta \in (0,1)$ such that

$$|A^{\alpha\beta}| \le \delta^{-1}, \quad \sum_{\alpha,\beta=1}^{d} \xi_\alpha \cdot A^{\alpha\beta}\xi_\beta \ge \delta \sum_{\alpha=1}^{d} |\xi_\alpha|^2$$

for any $\xi_\alpha \in \mathbb{R}^d$, $\alpha = 1, \ldots, d$.

$A^{\alpha\beta}$ are **measurable** in x_1. In particular, they may have jump discontinuities.

Thus, the system can be used to model, for example, the motion of two fluids with interfacial boundaries.

This type of coefficients was first considered by Krylov-Kim (2007) for non-divergence form elliptic equations with measurable coefficients.

Main results.

Theorem A. (D.-Kim, 2017) Let $q \in (1, \infty)$, and let Ω be either $\mathbb{R}^d$ or $\mathbb{R}^d_+$ and $A^{\alpha\beta} = A^{\alpha\beta}(x_1)$.
If $(u, p) \in W^1_q(\Omega)^d \times L_q(\Omega)$ satisfies

$$\begin{cases} \mathcal{L}u + \nabla p = D_\alpha f_\alpha & \text{in } \Omega, \\ \operatorname{div} u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \quad \text{in case } \Omega = \mathbb{R}^d_+, \end{cases}$$

where $f_\alpha, g \in L_q(\Omega)$, then we have that

$$\|Du\|_{L_q(\Omega)} + \|p\|_{L_q(\Omega)} \le N \left(\|f_\alpha\|_{L_q(\Omega)} + \|g\|_{L_q(\Omega)} \right).$$

We also considered Stokes system in a bounded Lipschitz Ω with a small Lipschitz constant.

In this case, we allow coefficients not only to be measurable locally in one direction (almost perpendicular to the boundary of the domain), but also to have small mean oscillations in the other directions.

Theorem B. (D.-Kim, 2017) Let $q \in (1, \infty)$, $K > 0$, and let Ω be bounded ($\operatorname{diam} \Omega \le K$). Then under the above assumptions, for $(u, p) \in W_q^1(\Omega)^d \times L_q(\Omega)$ satisfying $(p)_\Omega = 0$ and

$$\begin{cases} \mathcal{L}u + \nabla p = D_\alpha f_\alpha & \text{in } \Omega, \\ \operatorname{div} u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f_\alpha, g \in L_q(\Omega)$, we have that

$$\|Du\|_{L_q(\Omega)} + \|p\|_{L_q(\Omega)} \le N \left(\|f_\alpha\|_{L_q(\Omega)} + \|g\|_{L_q(\Omega)} \right).$$

Moreover, for $f_\alpha, g \in L_q(\Omega)$ with $(g)_\Omega = 0$, there exists a unique $(u, p) \in W_q^1(\Omega)^d \times L_q(\Omega)$ satisfying $(p)_\Omega = 0$ and the system.

Remark: For equations with VMO coefficients, such result was obtained by Choi-Lee (2016).

Idea of the proof.

1. The proof is based on pointwise sharp and maximal function estimates in the spirit of Krylov (2005). Such estimates rely on the $C^{1,\alpha}$ regularity of solutions to the homogeneous system.

2. Here, the main difficulty is that because the coefficients are measurable in x_1, it is impossible to obtain a Hölder estimate of the full gradient Du. To this end, instead of considering Du itself, we estimate certain linear combinations of Du and p:

$$D_{x'}u \quad \text{and} \quad U := A^{1\beta}D_\beta u + pe_1.$$

3. For this, we apply the **Caccioppoli** inequality for the stationary Stokes system:

$$\int_{B_r} |Du|^2 \, dx \le N(R-r)^2 \int_{B_R} |u|^2 \, dx,$$

as well as the following key lemma.

Lemma. Let $0 < r < R$, and let ℓ be a constant.

a. If $(u,p) \in W_2^1(B_R)^d \times L_2(B_R)$ satisfies the homogeneous system in B_R, then $D_{x'}p \in L_2(B_r)$ and

$$\int_{B_r} |D_{x'}p|^2 \, dx \le N(R-r)^{-2} \int_{B_R} |Du|^2 \, dx.$$

b. Similar boundary estimate holds.

4. Finally, in order to deal with the system in a Lipschitz domain, we apply a version of the Fefferman-Stein sharp function theorem for spaces of homogeneous type, which was recently proved in D.-Kim (2015), and employ a delicate cut-off argument, together with Hardy's inequality. The solvability follows from the **method of continuity**.

Weighted L_p estimates for stationary Stokes systems.

In a subsequent paper, we considered weighted L_p estimates for stationary Stokes systems in Reifenberg flat domains.

Muckenhoupt weights: for any $q \in (1, \infty)$, let $A_q = A_q(\Omega)$ be the set of all nonnegative $L_{1,\text{loc}}$ functions ω on Ω such that

$$[\omega]_{A_q} := \sup_{x_0 \in \Omega, r > 0} \left(\fint_{\Omega_r(x_0)} \omega(x)\, dx \right) \left(\fint_{\Omega_r(x_0)} \big(\omega(x)\big)^{-1/(q-1)}\, dx \right)^{q-1} < \infty.$$

Recall that $A_{q_1} \subset A_{q_2}$ for $q_1 < q_2$.

Theorem C. (D.-Kim, 2017) The result of Theorem A holds in the weighted space $L_{q,\omega}$ for any $\omega \in A_q$.

Theorem D. (D.-Kim, 2017) The result of Theorem B holds in the weighted space $L_{q,\omega}$ for any $\omega \in A_q$, when Ω is a bounded Reifenberg flat domains.

Remark of the proof: Instead of the method of mean oscillation estimates, we applied the level set argument in the spirit of Caffarelli and Peral, as the method continuity does not work here.

For this, we also derived a **reverse Hölder's inequality** for stationary Stokes systems:

There exists $\tilde{q} > 2$ such that

$$\left(|D\bar{u}|^{\tilde{q}}\right)^{1/\tilde{q}}_{B_r(x_0)} + \left(|\bar{p}|^{\tilde{q}}\right)^{1/\tilde{q}}_{B_r(x_0)} \\ \le N\left(|D\bar{u}|^2\right)^{1/2}_{B_{8r}(x_0)} + N\left(|\bar{p}|^2\right)^{1/2}_{B_{8r}(x_0)} + N\left(|\bar{f}_\alpha|^{\tilde{q}}\right)^{1/\tilde{q}}_{B_{8r}(x_0)} + \left(|\bar{g}|^{\tilde{q}}\right)^{1/\tilde{q}}_{B_{8r}(x_0)}$$

With Choi, we extended the results to Stokes systems with the conormal boundary condition.

Dini type estimates.

With better regularity of the coefficients and data, we obtain better regularity of solutions.

A partial Dini condition: We say that f is of *partially Dini mean oscillation with respect to* x' *in* B_4 if the function $\omega_{f,x'} : (0,1] \to [0,\infty)$ defined by

$$\omega_{f,x'}(r) := \sup_{x \in B_4} \fint_{B_r(x)} \left| f(y) - \fint_{B'_r(x')} f(y_1, z')\, dz' \right| dy$$

satisfies

$$\int_0^1 \frac{\omega_{f,x'}(r)}{r}\, dr < \infty.$$

Theorem E. (D.-Choi, 2018) Let $q_0 > 1$. Assume that $(u,p) \in W^{1,q_0}(B_6)^d \times L^{q_0}(B_6)$ is a weak solution of

$$\begin{cases} \mathcal{L}u + \nabla p = D_\alpha f_\alpha & \text{in } B_6, \\ \operatorname{div} u = g & \text{in } B_6, \end{cases}$$

where $f_1 \in L^\infty(B_6)^d$, $f_\alpha \in L^{q_0}(B_6)^d$, $\alpha \in \{2,\dots,d\}$, and $g \in L^\infty(B_6)$. Set

$$\hat{U} := A^{1\beta} D_\beta u + p e_1 - f_1.$$

(a) If $A^{\alpha\beta}$, f_α, and g are of partially Dini mean oscillation with respect to x' in B_4, then we have $(u,p) \in W^{1,\infty}(B_1)^d \times L^\infty(B_1)$ and

$$\hat{U}, D_\alpha u \in C(\overline{B_1})^d, \quad \alpha \in \{2,\dots,d\}.$$

(b) If it holds that $[A^{\alpha\beta}]_{C^{\gamma_0}_{x'}(B_6)} + [f_\alpha]_{C^{\gamma_0}_{x'}(B_6)} + [g]_{C^{\gamma_0}_{x'}(B_6)} < \infty$ for some $\gamma_0 \in (0,1)$, then we have

$$\hat{U}, D_\alpha u \in C^{\gamma_0}(\overline{B_1})^d, \quad \alpha \in \{2,\dots,d\}.$$

A similar result for elliptic systems with Dini mean oscillation coefficients was obtained by D.-Kim (2016).

Some remarks:

1. We also proved the corresponding boundary estimate in a half ball.

2. For general $C^{1,\text{Dini}}$ domains, we require the DMO condition instead of the partial DMO condition.

3. By using a duality argument, our results also hold when $q_0 = 1$ (in the spirit of Brezis).

II. A construction of Green matrices for stationary Stokes systems

Stokes systems have two types of Green functions.

One is a pair $(G, \Pi) = (G(x,y), \Pi(x,y))$, we call it *the Green function for the flow velocity*, satisfying

$$\begin{cases} \mathcal{L}G(\cdot,y) + \nabla\Pi(\cdot,y) = \delta_y I & \text{in } \Omega, \\ \operatorname{div} G(\cdot,y) = 0 & \text{in } \Omega, \\ G(\cdot,y) = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, G is a $d \times d$ matrix-valued function, Π is a $d \times 1$ vector-valued function.

The other one is a pair $(\mathcal{G}, \mathcal{P}) = (\mathcal{G}(x,y), \mathcal{P}(x,y))$, we call it *the Green function for the pressure*, satisfying

$$\begin{cases} \mathcal{L}\mathcal{G}(\cdot,y) + \nabla\mathcal{P}(\cdot,y) = 0 & \text{in } \Omega \setminus \{y\}, \\ \operatorname{div} \mathcal{G}(\cdot,y) = \delta_y - \dfrac{1}{|\Omega|} & \text{in } \Omega, \\ \mathcal{G}(\cdot,y) = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, $\mathcal{G}$ is a $d \times 1$ vector-valued function and $\mathcal{P}$ is a real-valued function

If there exist Green functions for the flow velocity and pressure, then the pair (u, p) given by

$$u(y) = \int_\Omega G(x,y)^\top f(x)\,dx, \quad p(y) = -\int_\Omega \mathcal{G}(x,y)\cdot f(x)\,dx$$

is a weak solution of the problem

$$\begin{cases} \mathcal{L}^* u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$.

The Green function for the flow velocity:

- For the classical Stokes system with the Laplace operator $\mathcal{L} = \Delta$: Ladyzhenskaya (1969), Maz'ya-Plamenevskiĭ (1983), and D. Mitrea-I. Mitrea (2011).
- Stokes systems with variable coefficients: Choi-Lee (2017) and Choi-Yang (2017).

There are relatively few results on Green functions for the pressure. For the classical Stokes system, see Maz'ya-Plamenevskiĭ (1983) and Maz'ya-Rossmann (2005).

Existence of the Green function for the pressure

Theorem F. (Choi-D., 2018) Let Ω be a bounded domain in $\mathbb{R}^d$. Suppose that the coefficients $A^{\alpha\beta}$ of $\mathcal{L}$ are of partially Dini mean oscillation with respect to x' in Ω Then there exists the Green function $(\mathcal{G}, \mathcal{P})$ for the pressure of $\mathcal{L}$ in Ω such that for any $y \in \Omega$, $\mathcal{G}(\cdot, y)$ is continuous in $\Omega \setminus \{y\}$ and

$$(\mathcal{G}(\cdot, y), \mathcal{P}(\cdot, y)) \in W^{1,\infty}_{\mathrm{loc}}(\Omega \setminus \{y\})^d \times L^\infty_{\mathrm{loc}}(\Omega \setminus \{y\}).$$

Moreover, for any $x, y \in \Omega$ with $0 < |x - y| \le d_y^*/2$, we have

$$|\mathcal{G}(x, y)| \le C|x - y|^{1-d}$$

and

$$\operatorname*{ess\,sup}_{B_{|x-y|/4}(x)} \big(|D\mathcal{G}(\cdot, y)| + |\mathcal{P}(\cdot, y)|\big) \le C|x - y|^{-d}$$

The same results hold if $\mathcal{L}$ is replaced by its adjoint operator $\mathcal{L}^*$

Global estimates.

Theorem G. (Choi-D., 2018) Let Ω be a bounded domain in $\mathbb{R}^d$ having $C^{1,\mathrm{Dini}}$ boundary. Suppose that the coefficients $A^{\alpha\beta}$ of $\mathcal{L}$ are of Dini mean oscillation in Ω. Let $(\mathcal{G}, \mathcal{P})$ be the Green function for the pressure of $\mathcal{L}$ constructed in Theorem F. Then for any $y \in \Omega$ and $r > 0$, we have

$$(\mathcal{G}(\cdot, y), \mathcal{P}(\cdot, y)) \in C^1(\overline{\Omega \setminus B_r(y)})^d \times C(\overline{\Omega \setminus B_r(y)}),$$

$$|\mathcal{G}(x,y)| \le C|x-y|^{1-d}, \quad |D_x\mathcal{G}(x,y)| + |\mathcal{P}(x,y)| \le C|x-y|^{-d}.$$

Moreover, if $(\mathcal{G}^*, \mathcal{P}^*)$ is the Green function for the pressure of $\mathcal{L}^*$, then for any $y \in \Omega$, there exists a measure zero set $N_y \subset \Omega$ containing y such that

$$\mathcal{P}(x,y) = \mathcal{P}^*(y,x) \quad \text{for all } x \in \Omega \setminus N_y.$$

For the proof, we adapted an argument of Grüter-Widman (1982). See also Hofmann-Kim (2007). We use the L^∞-estimates of (Du, p) in Theorem E.

A remark: Let (G, Π) and $(\mathcal{G}, \mathcal{P})$ be the Green functions for the flow velocity and the pressure of $\mathcal{L}$ in Ω, respectively. We define a $(d+1) \times (d+1)$ matrix-valued function by

$$\mathbf{G} = \begin{pmatrix} G^{11} & G^{12} & \dots & G^{1d} & -\mathcal{G}^1 \\ G^{21} & G^{22} & \dots & G^{2d} & -\mathcal{G}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ G^{d1} & G^{d2} & \dots & G^{dd} & -\mathcal{G}^d \\ \Pi^1 & \Pi^2 & \dots & \Pi^d & \mathcal{P} \end{pmatrix}.$$

Let (G^*, Π^*) and $(\mathcal{G}^*, \mathcal{P}^*)$ be the Green functions for the flow velocity and the pressure of $\mathcal{L}^*$ in Ω, respectively, and similarly define $\mathbf{G}^*$. Then

$$\mathbf{G}(x,y) = \mathbf{G}^*(y,x)^\top \quad \text{for all } y \in \Omega \text{ and a.e. } x \in \Omega.$$

III. L_p estimate for non-stationary Stokes systems

Known result for the classical non-stationary Stokes system: (Solonnikov (1964), Hu-Li-Wang (2014)) For $p \in (1, \infty)$,

$$\|D^2 u\|_{L_p(Q_{1/2})} \le N\|f\|_{L_p(Q_1)} + N\|u\|_{L_p(Q_1)}.$$

Notice that in contrast to the heat equation, we cannot estimate u_t by the right-hand side.

Counterexample by Serrin (1962): $u = \nabla H(x)g(t)$ and $p = -H(x)g'(t)$, where H is harmonic. Then (u, p) satisfies the Stokes system with $f = 0$. But $g \in C^1$ can be arbitrary, so it is impossible to get

$$\|u_t\|_{L_p(Q_{1/2})} \le N\|u\|_{L_p(Q_1)}.$$

The corresponding boundary estimate does **not** hold.

We are interested in such L_p estimate for **Stokes systems with variable coefficients** in divergence form:

$$u_t - D_i(a_{ij}D_j u) + \nabla p = \operatorname{div} f, \quad \operatorname{div} u = g.$$

We assume that $a_{ij} = b_{ij}(t,x) + d_{ij}(t,x)$, which satisfies the following boundedness and ellipticity conditions with ellipticity constant $\nu \in (0,1)$:

$$\nu|\xi|^2 \le a_{ij}\xi_i\xi_j, \quad |b_{ij}| \le \nu^{-1},$$

$$b_{ij} = b_{ji}, \quad d_{ij} \in L_{1,\text{loc}}, \quad d_{ij} = -d_{ji}, \quad \forall\, i,j \in \{1,2,\ldots,d\}.$$

In particular, a_{ij} can be **unbounded**.

We also consider the corresponding non-divergence form Stokes systems:

$$u_t - a_{ij}D_{ij}u + \nabla p = f, \quad \operatorname{div} u = g,$$

where $a_{ij} = b_{ij}$ is bounded and uniformly elliptic.

The VMO$_x$ (vanishing mean oscillation in x) condition

We impose the following VMO$_x$ introduced by Krylov, with constants $\delta \in (0, 1)$ and $\alpha_0 \in [1, \infty)$ to be determined later.

Assumption 1 (δ, α_0) *There exists $R_0 \in (0, 1/4)$ such that for any $(t_0, x_0) \in Q_{2/3}$ and $r \in (0, R_0)$,*

$$\fint_{Q_r(t_0,x_0)} |a_{ij}(t,x) - \bar{a}_{ij}(t)|^{\alpha_0} \, dx \, dt \leq \delta^{\alpha_0},$$

where $\delta \in (0, 1)$, $\alpha_0 \in [1, \infty)$, and $\bar{a}_{ij}(t)$ is the average of $a_{ij}(t, \cdot)$ in $B_r(x_0)$.

Function spaces.

For each $s, q \in [1, \infty)$ and parabolic cylinder $Q = \Gamma \times U \subset \mathbb{R} \times \mathbb{R}^d$, denote

$$\|u\|_{L_{s,q}(Q)} = \|u\|_{L_s(\Gamma; L_q(U))},$$

$$W^{1,2}_{s,q}(Q) = \left\{u : \ u, Du, D^2 u \in L_{s,q}(Q), \ u_t \in L_1(Q)\right\},$$

and denote $\mathbb{H}^{-1}_{s,q}(Q)$ the space consisting of all functions u satisfying

$$\left\{u = \operatorname{div} F + h \text{ in } Q : \|F\|_{L_{s,q}(Q)} + \|h\|_{L_{s,q}(Q)} < \infty\right\}.$$

Naturally, for any $u \in \mathbb{H}^{-1}_{s,q}(Q)$, we define

$$\|u\|_{\mathbb{H}^{-1}_{s,q}(Q)} = \inf\left\{\|F\|_{L_{s,q}(Q)} + \|h\|_{L_{s,q}(Q)} \,|\, u = \operatorname{div} F + h\right\},$$

and it is easy to see that $\mathbb{H}^{-1}_{s,q}(Q)$ is a Banach space. Finally, define

$$\mathcal{H}^1_{s,q}(Q) = \left\{u : \ u, Du \in L_{s,q}(Q), u_t \in \mathbb{H}^{-1}_{1,1}(Q)\right\}.$$

Main results.

Theorem 1 (D., Phan, 2018) Let $s, q \in (1, \infty)$, $\nu \in (0, 1)$, and $\alpha_0 \in \left(\frac{\min(s,q)}{\min(s,q)-1}, \infty\right)$. There exists $\delta = \delta(d, \nu, s, q, \alpha_0)$ such that the following statement holds. Under Assumption 1 (δ, α_0), if $(u, p) \in \mathcal{H}^1_{s,q}(Q_1)^d \times L_1(Q_1)$ is a weak solution to the Stokes system in Q_1, $f \in L_{s,q}(Q_1)^{d\times d}$, and $g \in L_{s,q}(Q_1)$, it holds that

$$\|Du\|_{L_{s,q}(Q_{1/2})} \le N\Big[\|f\|_{L_{s,q}(Q_1)} + \|g\|_{L_{s,q}(Q_1)}\Big] + N\|u\|_{L_{s,q}(Q_1)}.$$

Theorem 2 (D., Phan, 2018) Let $s, q \in (1, \infty)$ and $\nu \in (0, 1)$. There exists $\delta = \delta(d, \nu, s, q) \in (0, 1)$ such that the following statement holds. Under Assumption 1 $(\delta, 1)$, if $u \in W^{1,2}_{s,q}(Q_1)^d$ is a strong solution to the Stokes system in Q_1, $f \in L_{s,q}(Q_1)^{d\times d}$, and $Dg \in L_{s,q}(Q_1)^d$, then it follows that

$$\|D^2u\|_{L_{s,q}(Q_{1/2})} \le N\Big[\|f\|_{L_{s,q}(Q_1)} + \|Dg\|_{L_{s,q}(Q_1)}\Big] + N\|u\|_{L_{s,q}(Q_1)}.$$

Some Remarks.

1. When $q = s = 2$ and $g = 0$, we get a Caccioppoli type estimate for Stokes systems.
When $a_{ij} = \delta_{ij}$ and $f = g = 0$, such estimate was proved by B.J. Jin (2013) by using special test functions.
Note that the corresponding boundary Caccioppoli type estimate does **not** hold (Chang-Kang, 2018).

2. Our estimates do not contain the pressure term on the right-hand side, in contrast to the previous results, for instance, by Seregin. Thus our results are new even when the coefficients are constants.

3. The operators considered are more restrictive than those for the stationary Stokes systems.
The corresponding boundary estimate does **not** hold.

Applications to the Navier-Stokes equations.

Consider the Navier-Stokes equations (NSE)

$$u_t - \Delta u + (u \cdot \nabla) u + \nabla p = 0, \quad \operatorname{div} u = 0.$$

Let u be a Leray-Hopf weak solution of the NSE in Q_1.

For each $i, j = 1, 2, \ldots, d$, let d_{ij} be the solution of the equation

$$\begin{cases} \Delta d_{ij} = D_j u_i - D_i u_j & \text{in} \quad B_1 \\ d_{ij} = 0 & \text{on} \quad \partial B_1. \end{cases}$$

Let $[d_{ij}]_{B_\rho(x_0)}(t)$ be the average of d_{ij} with respect to x on $B_\rho(x_0)$.

As a corollary of Theorem 1, we obtain the following new ε-regularity criterion for the NSE.

Theorem 3. (D., Phan, 2018) Let $\alpha_0 \in (2(d+2)/(d+4), \infty)$. There exists $\varepsilon \in (0,1)$ sufficiently small depending only on the dimension d and α_0 such that, if u is a Leray-Hopf weak solution of NSE in Q_1 and

$$\sup_{z_0 \in Q_{2/3}} \sup_{\rho \in (0,R_0)} \left(\fint_{Q_\rho(z_0)} |d_{ij}(t,x) - [d_{ij}]_{B_\rho(x_0)}(t)|^{\alpha_0} \, dx \, dt \right)^{1/\alpha_0} \le \varepsilon,$$

for every $i, j = 1, 2, \ldots, d$ and for some $R_0 \in (0, 1/2)$, then u is smooth in $Q_{1/2}$.

We note that the parameter α_0 can be less than 2.

Idea of the proof. We rewrite the NSE into a Stokes system in divergence form with coefficients that have singular skew-symmetric part (d_{ij}). Then, we iteratively apply Theorem 1 and the Sobolev embedding theorem to successively improve the regularity of weak solutions.

Further applications.

Denote L_q^w to be the weak-L_s space, and $\mathcal{M}_{q,\beta}$ to be the Morrey space

$$\|f\|_{\mathcal{M}_{q,\beta}(B_1)} := \left(\sup_{x_0 \in \overline{B_1},\, r>0} r^{-\beta} \int_{B_r(x_0)\cap B_1} |f|^q \, dx \right)^{1/q}.$$

Corollary (Serrin type criteria). Assume that u is a Leray-Hopf weak solution of NSE in Q_1.

(i) Let $s, q \in (1, \infty]$ be such that $2/s + d/q = 1$. Suppose that $u \in L_s((-1,0); L_q^w(B_1))$ when $s < \infty$, or the $L_\infty((-1,0); L_d^w(B_1))$ norm of u is sufficiently small. Then, u is smooth in $Q_{1/2}$.

(ii) Let $s, q \in (1, \infty]$ be such that $2/s + d/q = 1$. Suppose that $u \in L_s^w((-1,0); L_q^w(B_1))$ with a sufficiently small norm. Then, u is smooth in $Q_{1/2}$.

(iii) Let $\alpha \in [0,1)$, $\beta \in [0,d)$, and $s, q \in (1,\infty)$ be constants satisfying

$$\frac{2\alpha}{s} + \frac{\beta}{q} = \frac{2}{s} + \frac{d}{q} - 1 (> 0), \quad \frac{1}{s} < \frac{1}{2} + \frac{1}{d+2}, \quad \text{and} \quad \frac{1}{q} < \frac{1}{2} + \frac{1}{d+2} + \frac{1}{d}.$$

Suppose that $u \in \mathcal{M}_{s,\alpha}((-1,0); \mathcal{M}_{q,\beta}(B_1))$ with a sufficiently small norm. Then, u is smooth in $Q_{1/2}$.

Some Remarks.

1. When $d = 3$, Corollary (i) recovers a result by Kozono (1998). When $d = 3$ and $q < \infty$, Corollary (ii) was obtained Kim and Kozono (2004). Our approach only uses linear estimates and is very different from these in those two papers.

2. We can take $q > 1$ and $s > 10/7$ in Corollary (iii) in the case when $d = 3$.

An outline of the proof of Theorem 1.

Step 1. We first consider Stokes systems with coefficients that only depends on t: $a_{ij} = a_{ij}(t)$.

Lemma 1. Let $q_0 \in (1, \infty)$, and $(u, p) \in \mathcal{H}^1_{q_0}(Q_1)^d \times L_1(Q_1)$ be a weak solution to the system in Q_1. Then we have

$$\|D^2 u\|_{L_{q_0}(Q_{1/2})} + \|Du\|_{L_{q_0}(Q_{1/2})} \le N(d, \nu, q_0)\|u - [u]_{B_1}(t)\|_{L_{q_0}(Q_1)},$$

where $[u]_{B_1}(t)$ is the average of $u(t, \cdot)$ in B_1.

Proof. By a mollification in x, we see that $\omega = \nabla \times u$ is a weak solution to the parabolic equation

$$\omega_t - D_i(a_{ij}(t)D_j\omega) = 0 \quad \text{in} \quad Q_1.$$

Because $(d_{ij}(t))_{n\times n}$ is skew-symmetric, ω is indeed a weak solution of

$$\omega_t - D_i(b_{ij}(t)D_j\omega) = 0 \quad \text{in} \quad Q_1.$$

Now we apply the local $\mathcal{H}_p^1$ estimate for linear parabolic equations to obtain

$$\|D\omega\|_{L_{q_0}(Q_{2/3})} \le N(d,\nu,q_0)\|\omega\|_{L_{q_0}(Q_{3/4})}. \tag{1}$$

Since u is divergence free, we have

$$\Delta u_i = -D_i \sum_{k=1}^d D_k u_k + \sum_{k=1}^d D_{kk} u_i = \sum_{k\neq i} D_k(D_k u_i - D_i u_k).$$

Thus by the local W_p^1 estimate for the Laplace operator,

$$\|Du\|_{L_{q_0}(Q_{1/2})} \le N\|\omega\|_{L_{q_0}(Q_{2/3})} + N\|u\|_{L_{q_0}(Q_{2/3})}.$$

Similarly,

$$\begin{aligned}
&\|D^2u\|_{L_{q_0}(Q_{1/2})} \le N\|D\omega\|_{L_{q_0}(Q_{2/3})} + N\|Du\|_{L_{q_0}(Q_{2/3})} \le N\|Du\|_{L_{q_0}(Q_{3/4})} \\
&\le \varepsilon\|D^2u\|_{L_{q_0}(Q_{3/4})} + N\varepsilon^{-1}\|u - [u]_{B_1}(t)\|_{L_{q_0}(Q_{3/4})}
\end{aligned}$$

for any $\varepsilon \in (0,1)$, where we used (1) in the 2nd inequality, and multiplicative inequalities in the last inequality.

It then follows from a standard iteration argument that

$$\|D^2 u\|_{L_{q_0}(Q_{1/2})} \le N\|u - [u]_{B_1}(t)\|_{L_{q_0}(Q_1)},$$

from which and multiplicative inequalities we obtain the desired estimate.

Lemma 2. Under the assumptions of Lemma 1, we have

$$\|\omega\|_{C^{1/2,1}(Q_{1/2})} \le N(d, \nu, q_0)\|\omega\|_{L_{q_0}(Q_1)},$$

where $\omega = \nabla \times u$.

Step 2. For the general case, we use a perturbation argument by estimating the mean oscillation of ω: Let $\mathcal{X} = Q_{2R/3}$ and $\omega^{\#}_{\text{dy}}$ of ω in $\mathcal{X}$ be the dyadic sharp function. Then,

$$\omega^{\#}_{\text{dy}}(z_0) \le N(d,\nu,q_0)\kappa^{-\frac{d+2}{q_0}}\mathcal{M}(I_{Q_{3R/4}}|f|^{q_0})^{1/q_0}(z_0) + N(d,q_0)\kappa\mathcal{M}(I_{Q_{3R/4}}|g|^{q_0})^{1/q_0}(z_0)$$
$$+ N(d,\nu,q_0)\Big(\kappa^{-\frac{d+2}{q_0}}\delta + \kappa\Big)\mathcal{M}(I_{Q_{3R/4}}|Du|^{q_1})^{1/q_1}(z_0) + N\kappa^{-d-2}(|\omega|)_{Q_{3R/4}}.$$

We also apply a local and mixed norm version of the Fefferman-Stein sharp function theorem recently established by D. and Kim (2015):

Lemma 3. For any $s, q \in (1,\infty)$, there exists a constant $N = N(d,s,q) > 0$ such that

$$\|f\|_{L_{s,q}(Q_R)} \le N\Big[\Big\|I_{Q_R} f^{\#}_{\text{dy}}\Big\|_{L_{s,q}(Q_R)} + R^{\frac{2}{s}+\frac{d}{q}-d-2}\|f\|_{L_1(Q_R)}\Big].$$

for any $R > 0$ and $f \in L_{s,q}(Q_R)$.

Thank you for your attention!